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Mathematical Reviews

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FOUNDATIONS

Hintikka, K. Jaakko J. Distributive normal forms in the calculus of predicates. Acta Philos. Fenn. 6, 71 pp. (1953).

Behmann [Math. Ann. 86, 163–229 (1922)] gave an illuminating decision method for the monadic predicate calculus (mpc): every formula is converted into its (uniquely defined) distributive normal form (dnf) by reducing the range of each quantifier as much as possible. An important by-product of this reduction is that every property definable in mpc can be obtained by Boolean operations from a set of mutually independent properties (complete description). The present author constructs an analogue of Behmann's reduction for the predicate calculus of first order. Now the dnf are not uniquely defined, and, of course, do not lead to a decision method except in special cases, e.g. the class discussed by von Wright [Soc. Sci. Fenn. Comment. Phys.-Math. 15, no. 10 (1950); MR 13, 521]. Though this class can be decided by a method due to Gödel [Monatsh. Math. Phys. 40, 433–443 (1933)], the use of dnf seems to have the following advantage: it gives a complete description of all the properties definable by means of expressions in von Wright's class. While no interesting class for formulae of the predicate calculus has yet been handled by the use of dnf beyond von Wright's class, the dnf seem to have applications in modal logic. *G. Kreisel* (Princeton, N. J.).

Hintikka, K. Jaakko J. Form and content in quantification theory. Acta Philos. Fenn. 8, 7–55 (1955).

Several interpretations of quantification theory (qt) are known: (i) the semantic one based on Gödel's completeness theorem, (ii) the intuitionist one based on his translation, (iii) the reviewer's no-counterexample-interpretation [J. Symb. Logic 16, 241–267 (1951), pp. 252–257; MR 14, 122] based on an extension of Herbrand's theorem. The present paper provides an interpretation of qt which is closely related to (iii). It is based on a new set of rules for qt (particularly P. 0–P. 4 on p. 44 and paragraph 4 on p. 54) which come about as follows: given a formula A of qt the author considers a method of constructing a set of formulae $S(A)$, $A \in S(A)$, which is closed with respect to the operations associated with the logical constants of qt [in particular, if $(Ex)P(x) \in S(A)$ a symbol c, not necessarily in A, is introduced such that $P(c) \in S(A)$]; at each (finite) stage of the construction of $S(A)$ there may be free choices, but it is always decidable whether the construction can be continued. A formula is called refutable if at some stage this construction cannot be continued, and provable if its negation is refutable. The author shows, practically from scratch, that this definition is equivalent to the usual formalizations of qt. If the construction of $S(\sim A)$ is called a counterexample to A, then a proof of A consists in showing that every proposed counterexample to A breaks down. The definition of 'counterexample' in (iii) takes a more familiar form, but is, in an obvious sense, equivalent to the author's. His treatment is neater than (iii) since it avoids the preliminary reduction to prenex form which is used in (iii). For any proof of A in orthodox qt, (iii) provides simple computable functionals which determine the stage at which the construction of $S(\sim A)$ breaks down in relation to the free choices made at the nth stage of the construction. The author's very clear argument in favour of a no-counterexample-interpretation of qt is further supported by the fact that the general decision methods for subclasses of qt [Ackermann, Solvable cases of the decision problem, North-Holland Publ. Co., Amsterdam, 1954; MR 16, 323] are unified by it. However, when qt is applied to arbitrary classes of axioms, as in some algebraic theories, then (i) is more natural.

The author's paper throws light on intuitionist mathematics, too; it is a clear example of Brouwer's fan theorem [Math. Ann. 97, 60–75 (1926)] which is central in intuitionist mathematics, and, like (iii), it gives a clearer meaning to the repeated negations which abound in intuitionist writing. *G. Kreisel* (Princeton, N. J.).

Asser, Günter. Eine semantische Charakterisierung der deduktiv abgeschlossenen Mengen des Prädikatenkalküls der ersten Stufe. Z. Math. Logik Grundlagen Math. 1, 3–28 (1955).

The author considers notions of logical derivation which satisfy certain closure conditions and which may include also various familiar rules of inference of the predicate calculus, in particular, the rules of substitution for individual and predicate variables. A set of sentences (in the most important case—within the lower predicate calculus) is "deductively closed" if A is closed with respect to the given notion of derivation. A "predicate logical matrix" μ is a system which consists of a set of "truth values" M; a subset M^* of M ("designated truth values"); functions of negation, conjunction, implication, etc., with one or two arguments in M and taking values in M; and of two quantification functions (existential and universal) which are set functions on the subsets of M and take values in M. By means of a matrix μ as described, we can define the truth value of a sentence S with respect to a structure (model) J if the free individual and predicate variables of S are mapped on elements and relations of J such that the truth values of the atomic sentences which involve these elements and relations are given a priori. The truth value of S is then defined by means of μ in much the same way as the ordinary truth value of a sentence ("true" or "false") is defined by means of the standard semantic interpretation of the propositional functions and quantifiers of the lower predicate calculus. S is "valid" in J if it obtains a designated truth value for all possible mappings of the variables of S on the elements and relations of J. The principal result of the paper is that for every set of sentences X which is deductively closed, there exist a matrix μ and a model J as described above such that X coincides with the set of sentences which are valid in J. In this context, the structure J contains a

relation of order n for every positive integer n, but in general not all relations which can be defined on the set of elements of J will appear in the structure J. More precise conditions can be laid down for the matrix μ depending on the notion of derivation which is under consideration.

The above result is the analogue of a theorem of Lindenbaum's for the calculus of propositions. One may also expect that there is some connection between the present theory and the theory of Łoś as described by Mostowski [Uspehi Mat. Nauk (N.S.) 9, no. 3(61), 3–38 (1954); MR 16, 552].

The author uses his result in order to prove that the two sentences $\sim F$ and $\sim G$ are not derivable from one another, where F and G are two particular sentences which are identically true for finite but not for infinite models [see Hilbert and Bernays, Grundlagen der Mathematik, vol. I, Springer, Berlin, 1934, pp. 123–124]. This result has been obtained previously by G. Hasenjaeger [J. Symb. Logic 15, 273–276 (1950); MR 12, 578]. *A. Robinson* (Toronto, Ont.).

On order

✱Robinson, Abraham. Théorie métamathématique des ideaux. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1955. 186 pp.

In dem vorliegenden Buch, das die Untersuchungen des Verf. [On the metamathematics of algebra, North-Holland Publ. Co., Amsterdam, 1951; MR 13, 715] fortsetzt, haben wir es nicht mit Metamathematik (Mm.) im Sinne von Hilbert, sondern mit einer von Tarski und dem Verf. begründeten neuartigen Anwendung der modernen Logik auf die Mathematik zu tun. Gegenstand dieser Mm. sind axiomatische Theorien, deren Axiomensysteme im elementaren Logikkalkül (mit Quantoren) darstellbar sind. Der Gödelsche Vollständigkeitssatz sagt aus, dass eine Menge von Aussagen einer solchen Theorie stets ein Modell hat, wenn aus keiner endlichen Teilmenge ein Widerspruch abzuleiten ist. Dieser Satz hat ein Analogon in der Theorie der algebraischen Gleichungen: eine Menge von algebraischen Gleichungen (mit Koeffizienten aus einem Körper K) hat allemal eine Lösung in einem Oberkörper von K, wenn aus keiner endlicher Teilmenge die 1 linear zu kombinieren ist. Verf. stellt dar, wie dieses und viele andere Ergebnisse über algebraische Mannigfaltigkeiten als Spezialisierungen mm. Sätze erhalten werden können. Manche neuen Resultate, von denen nicht bekannt ist, wie sie mit den bisherigen Methoden zu erhalten sind, ergeben sich aus der neuartigen Betrachtungsweise. Es handelt sich also um eine Bereicherung der axiomatischen Methode, die durch das Ineinandergreifen von Algebra und Logikkalkül besonders reizvoll ist. Diese neue Mm. unterscheidet sich von der Hilbertschen noch dadurch, dass sie sich methodisch nicht auf das Finite beschränkt, sondern die naiv-mengentheoretischen Begriffsbildungen benutz—wozu sie, da sie ja keine Grundlagenforschung mehr sein will, natürlich berechtigt ist. In den ersten fünf Kapiteln werden nach vorbereitenden mathematischen Betrachtungen über topologische Räume und geordnete Mengen ausführlich die Grundlagen der neuen Mm. dargestellt: die Tarski-Systeme, der Logikkalkül mit dem Gödelschen Satz und algebraischen Strukturen (das sind Modelle elementarer Axiomensysteme mit einer Gleichheitsrelation). Die Darstellung enthält hier gegenüber dem früheren Buch viele Neuerungen.

In den nächsten beiden Kapiteln wird die Theorie der "Mannigfaltigkeiten von Strukturen" entwickelt, die die übliche Theorie der algebraischen Mannigfaltigkeiten verallgemeinert: es besteht ein Verbandsisomorphismus zwischen den Idealen von Aussagen einer elementaren Theorie und den Mannigfaltigkeiten von Strukturen. Als Spezialisierungen werden Sätze über kommutative und nicht-kommutative Ringe, Gruppen, Halbgruppen, Polynomringe und Ringe mit Derivation abgeleitet. Das letzte Kapitel behandelt die besonders interessanten "théorèmes de transfert". Beispiel: eine elementare Aussage der Körpertheorie, die für alle Körper der Charakteristik 0 gilt, gilt auch für alle Körper der Charakteristik $p > p_0$ (für geeignetes p_0). Die Fülle der Beispiele, die das Buch bringt, zeigt deutlich, wie sehr der Algebra die Einbeziehung dieser Mm. in ihre Forschungsmethoden empfohlen werden kann. *P. Lorenzen* (Bonn).

Henkin, Leon. On a theorem of Vaught. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17, 326–328 (1955).

Using a theorem (VL) of Vaught [same Proc. 57, 467–472 (1954); MR 16, 208] and Łos [Colloq. Math. 3, 58–62 (1954); MR 15, 845], the author shows that a theory T with standard formalization is decidable provided (a) for some infinite cardinal c, any two models of T of power c are isomorphic, (b) T is finitely axiomatizable. VL contains the additional hypothesis that all models of T are infinite, but, instead of (b), requires T only to be (recursively) axiomatizable. The author first adds to T sentences B_n ($n=2, 3, \cdots$) expressing that there exist $\geqq n$ distinct individuals, and applies VL to this enlarged theory. If a first-order sentence σ can be proved in the enlarged theory, then, by the completeness theorem, σ can also be proved in $T_U B_N$ for a fixed N. Since T is finitely axiomatizable one can effectively determine all the models of T with $<N$ elements, and can decide whether σ follows from $T_U \sim B_N$. The author mentions work of Ehrenfeucht which shows that 'finitely' is necessary in (b), and observes that the theory of Abelian groups in which every non-zero element is of order p (p prime and fixed) is covered by this theorem, but not by VL. *G. Kreisel* (Princeton, N. J.).

Beneš, Václav Edvard. On the consistency of an axiom of enumerability. J. Symb. Logic 20, 29–30 (1955).

Verf. bemerkt im Anschluss an eine Arbeit von H. Wang [J. Symb. Logic 15, 25–32 (1950); MR 11, 636] dass für jede wf. Axiomatisierung S der Mengenlehre die Erweiterung S' aller Klassen von Mengen von S noch wf. bleibt, wenn axiomatisch eine Ordnung vom Typ ω für die Allklasse gefordert wird. *P. Lorenzen* (Bonn).

Shimony, Abner. Coherence and the axioms of confirmation. J. Symb. Logic 20, 1–28 (1955).

The author calls the non-frequentist concept of probability 'confirmation', following R. Carnap [Logical foundations of probability, Univ. of Chicago Press, 1950; MR 12, 664]. This term has been criticised by K. R. Popper [British J. Philos. Sci. 5, 143–149 (1954); MR 16, 376]. Perhaps the term 'credibility' is preferable, as used, for example, by F. Y. Edgeworth [Encyclopaedia Britannica, v. 22, 11th ed., Cambridge, 1911, p. 377]. The author attempts to show that the usual axioms of credibility are analytic, by appeal to a principle of coherence (consistency). This principle is stated in terms of possible sets of bets. The method of proof resembles that used by F. P. Ramsey [The foundations of mathematics, Paul-Trench-Trubner, London, 1931, Chapters 7 and 8], and B. de Finetti [Fund. Math. 17, 298–329 (1931)], but is claimed to be improved by a more explicit formulation of the principle of coherence. The author's intuition boggles at the inclusion of an axiom of complete

additivity, since it implies that a set of mutually exclusive and exhaustive propositions cannot be equiprobable. The reviewer is not worried by this implication [see, e.g., Probability and the weighing of evidence, Griffin, London, Hafner, New York, 1950, p. 55; MR **12**, 837]. Finally there is a treatment of the comparative axioms of B. O. Koopman [Ann. of Math. (2) **41**, 269–292 (1940); MR **1**, 245].
I. J. Good (Cheltenham).

Bar-Hillel, Yehoshua. An examination of information theory. Philos. Sci. **22**, 86–105 (1955).
This paper is an expository outline for the philosophically interested in which the author aims to clarify certain basic misconceptions which have crept into the foundations of Information Theory. The author discusses the similarities and differences between information theory, communication theory, and the information calculus. It is the latter which is applied in psychology, linguistics, sociology, anthropology, physics, etc. The author concludes that no direct impact on Semantics is to be expected from information theory but that the information calculus may become a powerful tool in (Inductive) Semantics.
S. Kullback.

Martin, Gottfried. Methodische Probleme der Metaphysik der Zahl. Studium Gen. **6**, 610–616 (1953).

ALGEBRA

***van der Waerden, B. L. Algebra. Teil I. 4te Aufl.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Bd. XXXIII. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. viii+292 pp. DM 29.60.
This edition differs from the 3rd [1950; MR **12**, 236] in a change of title, correction of misprints, and minor changes and additions.

***Queysanne, M., et Delachet, A. L'algèbre moderne.** Presses Universitaires de France, Paris, 1955. 136 pp.
This book gives a brief survey of modern algebra, including its evolution from classical algebra. *R. E. Johnson.*

Clatworthy, Willard H. Partially balanced incomplete block designs with two associate classes and two treatments per block. J. Res. Nat. Bur. Standards **54**, 177–190 (1955).
A design is called connected if any two blocks of the design can be connected by a chain of blocks such that any two adjacent blocks of the chain have a treatment in common. The author considers only connected designs and tabulates all parameter combinations with $r \leqq 10$, for which according to present knowledge a solution may exist. Construction methods are given for all but four of these parameter combinations. The four for which no decision has been reached have block-sizes 105, 130, 175, 280 and their construction seems at present only of academic interest.
H. B. Mann (Columbus, Ohio).

Sprott, D. A. Some series of partially balanced incomplete block designs. Canad. J. Math. **7**, 369–381 (1955).
Generalizing a construction method of R. C. Bose and K. R. Nair [Sankhyā **4**, 337–372 (1939)] the author constructs 4 series of partially balanced incomplete block designs. The conditions on the parameters of these designs are too complicated to be quoted in a review. *H. B. Mann.*

Bruck, R. H. Difference sets in a finite group. Trans. Amer. Math. Soc. **78**, 464–481 (1955).
A λ-plane (or symmetric block design) is a set of v lines and v points such that every line contains k points and any pair of lines intersects in λ points. Suppose that G is a finite group transitive and regular on the points of a λ-plane preserving incidences. Then labeling an arbitrary point of the plane as the identity of G, we assign as a label to another point that group element which maps the identity point onto it. The set D of k elements on a line is called a difference set in the group G and has the properties that each $x \neq 1$ of G can be expressed in exactly λ ways in the form $x = d_i d_j^{-1}$ with d_i, d_j from D and also in exactly λ ways in the form $x = d_i^{-1} d_j$. Conversely these two properties are equivalent to each other and such a subset D of a finite group yields a λ-plane whose points are the elements of G and whose lines are the sets Dx as x ranges over G. This is a generalization of work by the reviewer and H. J. Ryser [Canad. J. Math. **3**, 495–502 (1951); MR **13**, 312] which treated the case in which G is cyclic.
In the group ring H of G over the rational field let Δ be the sum of the elements of D and S the sum of all elements. Also let $u \rightarrow u^*$ be the anti-isomorphism of H onto itself determined by mapping the elements of G onto their inverses. Then $\Delta\Delta^* = \Delta^*\Delta = n + \lambda S$ is the basic relation for Δ where, as usual $n = k - \lambda$. A multiplier is defined as an automorphism θ of G such that $\Delta\theta = a\Delta b$, where a and b are some elements of G. It is proved that whenever G is abelian a prime $p > \lambda$ such that $(p, v) = 1$, $p \mid n$, will yield a non-trivial multiplier, this being given by $x \rightarrow x^p$ for $x \in G$. This result is generalized and a variety of results on multipliers and subsystems are found. The case $\lambda = 1$ of finite projective planes is examined carefully. Unfortunately this has not led to any new planes. *Marshall Hall, Jr.*

Borsellino, Antonio. Su alcune identità che intervengono nella risoluzione della equazione di Chandrasekhar e Münch. Mem. Soc. Astr. Ital. (N.S.) **26**, 189–192 (1955).
Let D_h denote the determinant

$$D_h = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ q_0 & q_1 & q_2 & \cdots & q_n \\ q_0^2 & q_1^2 & q_2^2 & \cdots & q_n^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ q_0^{h-1} & q_1^{h-1} & q_2^{h-1} & \cdots & q_n^{h-1} \\ q_0^h & q_1^h & q_2^h & \cdots & q_n^h \end{vmatrix}$$

and $F_h = D_h/D_n$. Clearly $F_h = 0$ for $h < n$ and $F_n = 1$. For $h > n$ the identity

$$(*) \qquad F_{n+m} = w_m \quad (m \geqq 0),$$

is established, where w_m is Wronski's aleph function of degree m in the q's. The author shows that the relation (*) enables a simple derivation of the many identities which occur in the solution of the integral equation in the theory of the fluctuation in brightness of the Milky Way [S. Chandrasekhar and G. Münch, Astrophys. J. **112**, 380–392 (1950); MR **12**, 644; A. Ramakrishnan and P. M. Mathews, ibid. **119**, 81–90 (1954); MR **15**, 716].
S. Chandrasekhar (Williams Bay, Wis.).

Penrose, R. A generalized inverse for matrices. Proc. Cambridge Philos. Soc. **51**, 406–413 (1955).

A generalized inverse (g.i.) is introduced for arbitrary (possibly rectangular) matrices A with complex elements. It is the unique solution X of the four equations $AXA=X$, $XAX=X$, $(AX)^*=AX$, $(XA)^*=XA$. [Abstract rings with inverses satisfying $axa=a$ had been studied by v. Neumann, Proc. Nat. Acad. Sci. U. S. A. **22**, 707–713 (1936).] The g.i. is used to formulate a necessary and sufficient condition for the solvability of the matrix equation $AXB=C$ and of the system $AX=C$, $XB=D$ and to give an explicit solution. They are further used to give explicit expressions for the principal idempotents of a matrix and for a new type of spectral decomposition which allows the g.i. of A to be expressed in a simple manner in terms of the principal idempotents even for non-normal matrices.

O. Taussky-Todd (Washington, D. C.).

Eršov, A. P. On a method of inversion of matrices. Dokl. Akad. Nauk SSSR (N.S.) **100**, 209–211 (1955). (Russian)

The following direct method of inverting a matrix A is derived. Let $S^{(0)}=A-E$ where $E=$identity. A sequence $S^{(1)'}, S^{(1)}, \cdots, S^{(n)'}, S^{(n)}$ is defined as follows:

$$s_{ij}^{(m)'}=\begin{cases} s_{ij}^{(m-1)} & (i\neq m) \\ \delta_{mj} & (i=m) \end{cases} \quad (\text{all } i, j;\ m=1, \cdots, n);$$

$$s_{ij}^{(m)}=s_{ij}^{(m)'}-s_{im}^{(m)'}s_{mj}^{(m-1)}[1+s_{mm}^{(m-1)}]^{-1}.$$

Then $S^{(n)}=A^{-1}$. Some other properties are proved, and a connection is shown with elimination. *G. E. Forsythe.*

Kotelyanskiĭ, D. M. On the influence of Gauss's transformation on the spectrum of matrices. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 117–121 (1955). (Russian)

In Gauss's elimination process the real matrix

$$A=\|a_{ik}\|_1^n \quad (a_{11}>0)$$

is transformed into $B=\|b_{ik}\|$, where

$$b_{i1}=\delta_{i1}a_{i1};\quad b_{ik}=a_{11}^{-1}(a_{11}a_{ik}-a_{i1}a_{1k}) \quad (k>1).$$

Let $A_{11}=\|a_{ik}\|_2^n$. Let $A(\lambda)$ be the characteristic polynomial $|\lambda I-A|$, etc. In this posthumous paper the author proves that $B(\lambda)=a_{11}^{-1}[\lambda A_{11}(\lambda)-A(\lambda)]$, and uses this to discuss the spectra of A and B. Most detail is possible when the roots λ_i of $A(\lambda)$ and the roots μ_i of $A_{11}(\lambda)$ are real and separate each other $(\lambda_1\leqq\mu_1\leqq\lambda_2\cdots\leqq\mu_{n-1}\leqq\lambda_n)$, as occurs for hermitian matrices and for completely nonnegative matrices [F. R. Gantmaher and M. G. Kreĭn, Oscillation matrices and kernels and small vibrations of mechanical systems, 2d ed., Gostehizdat, Moscow-Leningrad, 1950; MR **14**, 178]. The following results (and a little more) are then proved: (1) in each closed interval $[\mu_i, \mu_{i+1}]$ lies one root of $B(\lambda)$; (2) in each closed interval $[\lambda_i, \lambda_{i+1}]$ lying on one side of 0 there is one root of $B(\lambda)$. Moreover, if $\lambda_1<0$, one root of $B(\lambda)$ lies in the closed interval $(-\infty, \lambda_1]$. The principal tool here is a pencil of matrices $(0\leqq t\leqq 1)$ reducing to A for $t=0$ and to B for $t=1$. *G. E. Forsythe* (New York, N. Y.).

On order

✱Cotlar, Mischa. The problem of moments and the theory of Hermitian operators. Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 71–85. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

A historical survey of the theory of moments including recent Russian work. *D. C. Kleinecke.*

Mahajani, G. S., and Thiruvenkatachar, V. R. Remarks on a problem in symmetric functions. Proc. Indian Acad. Sci. Sect. A. **41**, 225–230 (1955).

The problem discussed in an earlier paper [Mahajani, Thiruvenkatachar, and Thawani, same Proc. **35**, 211–223 (1952); MR **14**, 8] is "reformulated" as follows: Let $\alpha_{i-1}\leqq x_i\leqq\alpha_i$ $(i=1, \cdots, n)$, and let k_r $(r=1, \cdots, n)$ be the elementary symmetric functions of the x_i. Given the α_i and all but one of the k_r, determine the x_i so that the remaining k_r is an extremum. This is said to offer "a natural approach" to the original problem "but it seems difficult to carry out this programme in general". *A. Erdélyi.*

On order

✱Ślebodziński, Władysław. Formes extérieures et leurs applications. Vol. I. Monografie Matematyczne, Tom XXXI. Państwowe Wydawnictwo Naukowe, Warszawa, 1954. vi+154 pp. zł. 24.60.

This is the first of two volumes on the subject of exterior forms and their applications. It is devoted to an exposition of the algebra of exterior forms, whereas the second volume, as yet unpublished, will be concerned with questions of analysis. Volume I comprises four chapters. The first covers the basic definitions and algebraic notions, including that of the derivative and rank of an exterior form. A section on divisibility is also included. Chapter II is on exterior equations and is the algebraic preparation for exterior differential systems. Regular and singular solutions and the characters of a system are all discussed as are characteristic elements and the class of a system. Special attention is given to systems containing only linear and quadratic forms. Chapter III gives an exposition of symplectic space and the symplectic group (as the group of linear transformations leaving invariant a quadratic exterior form) and includes a discussion of the invariants of an exterior quadratic form under the symplectic group. Chapter IV gives three further applications of exterior forms: the fundamental theorems of determinant theory, infinitesimal rotations, and linear complexes. A short bibliography is included here and a complete bibliography of the subject is promised in the second volume. The treatment throughout is in terms of classical tensor calculus and some knowledge of this is assumed on the part of the reader. The book is compact and readable and should be of interest to those who wish to study the theory of exterior differential systems as developed by E. Cartan.

W. M. Boothby (Evanston, Ill.).

Abstract Algebra

✱Séminaire d'algèbre et de théorie des nombres dirigé par A. Châtelet et P. Dubreil, 1953/1954. Vingt conférences avec pagination individuelle par F. Châtelet, R. Croisot, J. Guérindon, P. Jaffard, M. Lazard, L. Lesieur, J. Petresco, G. Poitou, J. Riguet, P. Samuel, Mlle. M. Teissier, R. Thibault, et G. Thierrin. Faculté des Sciences de Paris, Paris, 1954. 189 pp. (polycopiées). (Order from Secrétariat mathématique, 11 rue Pierre Curie, Paris 5.)

These multilithed and paper-bound notes are a collection of papers given at "Faculté des Sciences de Paris" during the academic year 1953–1954.

In part I are lectures on various topics in algebra, many of which have appeared (or will appear) in print and it will suffice to list these: L. Lesieur, four lectures on geometric and topological lattices [Dubreil-Jacotin, Lesieur, Croisot,

Leçons sur la théorie des treillis des structures algébriques ordonnées . . . , Gauthier-Villars, Paris, 1953; MR **15**, 279; Lesieur, C. R. Acad. Sci. Paris **238**, 1464–1466 (1954); MR **16**, 329]; M. Lazard, "The identity of Hall and typical sequences" [ibid. **236**, 36–38 (1953); MR **14**, 617]; J. Petresco, "On commutators"; G. Poitou, "Diophantine approximations and modular groups"; F. Châtelet, "Rational points on cubic surfaces"; P. Samuel, "The Lemma of Hensel"; J. Guerindon, "On the maximal chains of ideals in a ring"; P. Jaffard, "Extensions of ordered groups".

Part II consists of reports in a seminar on semi-groups [see Thierrin, Acad. Roy. Belg. Bull. Cl. Sci. (5) **39**, 942–947 (1953); MR **15**, 680; Croisot, Ann. Sci. Ecole Norm. Sup. (3) **70**, 361–379 (1953); Bull. Soc. Math. France **82**, 161–194 (1954); C. R. Acad. Sci. Paris **239**, 845–847 (1954); MR **15**, 680; **16**, 215]. In addition, there are expositions by R. Thibault on groups homomorphic to a semi-groups and embedding semi-groups in groups; a lecture by J. Riguet on the recent work of Malcev, Vagner and Lyapin on the representation of semi-groups; a lecture by Mlle. M. Teissier on completely simple semi-groups. *L. J. Paige.*

Nishigori, Noboru. A note on lattice segment. J. Sci. Hiroshima Univ. Ser. A. **18**, 123–127 (1954).

Given two elements a and b of a lattice L, the set of all elements $x \in L$ with $ab \leqq x \leqq a+b$ is called the segment joining a and b. The author gives an axiom system for lattices with a zero element, using the notion of a segment as an undefined term. *B. Jónsson* (Berkeley, Calif.).

Hughes, N. J. S. Refinement and uniqueness theorems for the decompositions of algebraic systems with a regularity condition. J. London Math. Soc. **30**, 259–273 (1955).

A Σ-set is a set with one or more operations, and an S-system is a set of Σ-sets together with a set of functions f, each of which maps a subset of a Cartesian product of two or more Σ-sets into a Σ-set. Certain additional conditions of a very general nature are imposed upon the Σ-sets and upon the functions f. By a decomposition of an S-system $\mathfrak{S}$ is meant, roughly speaking, a system of simultaneous decompositions of all the Σ-sets in $\mathfrak{S}$, which are "consistent" under all the functions f. The principal result states that, under suitable conditions on the Σ-set H in $\mathfrak{S}$, any two decompositions of H which are induced by decompositions of $\mathfrak{S}$ possess a common refinement. Applied to rings, groups, and partially ordered sets with a zero element, this yields both new and known refinement theorems. *B. Jónsson.*

***Schmidt, Jürgen. Einige grundlegende Begriffe und Sätze aus der Theorie der Hüllenoperatoren.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 21–48. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

The author deals with sets in which a closure operation is defined i.e. the family of closed sets form an intersection ring. Such a system is called algebraic if it is the closure under some operation or relation involving only a finite number of elements. This is equivalent to the finiteness condition that an element belongs to the closure $C(A)$ of a set A only if it belongs to the closure of a finite subset of A and this in turn is equivalent to the inductive property that the sum of every ascending chain of closed sets is closed. The author points out the connection with the system used in Tarski's metamathematics.

For algebraic closure systems one has the analogues of the algebraic decomposition theorems, for instance, the theorem of McCoy-Fuchs that every closed set is the intersection of absolutely irreducible such sets.

The author also points out an analogue to the Frattini-Neumann results about the intersection of maximal closed sets. Finally algebraic systems with a symmetric relation are considered. Here orthogonality can be defined and certain results about maximal orthogonal sets to a given one are derived. *O. Ore* (New Haven, Conn.).

***Kerstan, Johannes. Elementfreie Begründung der allgemeinen Ideal- und Modultheorie.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 49–57. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

The author discusses the question of how far the ideal theory in rings may be founded exclusively upon lattice-theoretical axioms, in particular, the requirements to obtain the theorem in Noether rings that every irreducible ideal shall be primary. *O. Ore* (New Haven, Conn.).

Artzy, Rafael. On loops with a special property. Proc. Amer. Math. Soc. **6**, 448–453 (1955).

Let G be a loop with property π: $xyx'=y$, where x' is the right inverse of x. The author defines a cycle of inverses of length n as a sequence of elements $x_1, \cdots, x_n$, where $x_{k+1}=x_k'$. If n is 1 or 2, the right and left inverses are identical. If loop G consists only of the unit element u and m cycles of the same length n, n is necessarily a factor of $2m$. Let G have a cycle of length $n>2$; then there will be another cycle whose length is a factor of n; that implies the existence of sequences of cycles such that every length is a factor of the length of the successor. The unit element and elements of cycles of length n and any factor of n form a subloop. Therefore we obtain a sequence of subloops such that each subloop contains its predecessors. If two loops have identical numbers of cycles of every length, they are not necessarily isomorphic. Furthermore, the author shows there are infinitely many nonisomorphic loops of infinite order with property π consisting only of u and one cycle of infinite length. If an infinite loop with property π contains finite cycles, they form, together with u, a subloop. The author gives multiplication tables for three loops with more than one sequence of cycles. *H. Orlik-Pflugfelder.*

Bruck, R. H. Analogues of the ring of rational integers. Proc. Amer. Math. Soc. **6**, 50–58 (1955).

A system $(R, +, \cdot)$ with two single-valued binary operations is called a right neoring if (1) $(R, +)$ is a loop with zero 0, (2) R is closed under $(\cdot)$ and $x0=0$, $x \in R$, (3) $(x+y)z=xz+yz$; and R is called neoring if in addition (4) $x(y+z)=xy+xz$ is valid. If, further, R is commutative and associative (with respect to multiplication), contains at least two elements, and no nonzero divisors of zero, then R is called an integral neodomain. If, in addition, division is performable in R, then R is termed neofield. The author first remarks that an integral neodomain can be imbedded into a quotient neofield. To obtain a deeper insight into the structure of neorings he considers a free loop $(A, +)$ on one free generator 1. For any $x \in A$ there is a unique endomorphism $\varphi(x)$ of $(A, +)$ such that $1\varphi(x)=x$ and he defines a multiplication in A by setting $yx=y\varphi(x)$. Thus he obtains an associative neoring $(A, +, \cdot)$ which he terms universal in virtue of the following result: Let $(R, +, \cdot)$ be a right neoring with left identity 1 such that 1 generates the additive loop $(R, +)$. Let τ be the (always available) homomorphism of $(A, +)$ upon $(R, +)$ such that $1\tau=1$. Then τ

is a homomorphism of $(A, +, \cdot)$ upon $(R, +, \cdot)$. By using a fundamental property of free loops he further shows that any free loop $(F, +)$ of finite or of countable rank can be imbedded in at least one associative right neoring $(F, +, \cdot)$ satisfying both multiplicative cancellation laws and the relation $FG \subseteq G$ for any subloop $(G, +)$ of $(F, +)$. Another result gives a necessary and sufficient condition that a subloop of a right neoring R be a kernel of a homomorphism of R upon a right neoring. Finally, by using a recent work of R. Artzy [see the paper reviewed above], he exhibits infinitely many nonisomorphic minimal neofields of countable order. Various other results and some unsolved problems are stated. *J. Levitzki* (Jerusalem).

McCoy, Neal H. Subdirect sum representations of prime rings. Duke Math. J. **22**, 357–363 (1955).

A prime ring R is a ring in which the zero ideal is a prime ideal. The author notes that with R also the polynomial domain $R[x]$ is prime. His object is to study the connection between the proper representations of a prime ring R as a subdirect sum of prime rings and similar representations of $R[x]$. His results are generalizations of those obtained by W. Krull [Math. Z. **52**, 810–826 (1950); MR **12**, 155] for the case of integral domains. He proves that if a countable prime ring R with infinite center has a proper representation as a subdirect sum of a countable number of prime rings, then the polynomial domain $R[x]$ has a proper representation as a subdirect sum of these same prime rings. The case of a finite center is more complicated. For any ring R denote by $N(R)$ the cardinal number of the center of R. It follows that if the prime ring R is isomorphic to a subdirect sum of prime rings R_α and the $N(R_\alpha)$ are bounded, then $N(R)$ is finite. Since the center of $R[x]$ is infinite, it follows that under these conditions $R[x]$ can not be isomorphic to a subdirect sum of the same rings R_α. He finally proves that if R is a countable prime ring with finite nonzero center, which has a proper representation as a subdirect sum of a countable number of prime rings R_i $(i=1, 2, \cdots)$ and if P_i $(i=1, 2, \cdots)$ are the associated nonzero prime ideals in R, then: 1) if there exists a sequence P_{i_k} $(k=1, 2, \cdots)$ with zero intersection such that $N(R_{i_k})>k$ $(k=1, 2, \cdots)$, $R[x]$ also has a proper representation as a subdirect sum of the same prime rings R_i $(i=1, 2, \cdots)$; 2) if $R[x]$ is not isomorphic to a subdirect sum of the same rings R_i $(i=1, 2, \cdots)$, there exists a sequence P_{i_k} $(k=1, 2, \cdots)$ with zero intersection such that the $N(R_{i_k})$ are bounded. The author remarks that his method may be used to extend the greater part of Krull's remaining results to the present more general setting. *J. Levitzki* (Jerusalem).

Szele, Tibor. Geometrical proof of two structure theorems of ring theory. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 49–85 (1954). (Hungarian)

Expanded version of a paper in Acta Math. Acad. Sci. Hungar. **5**, 101–107 (1954); MR **16**, 213.

Hebroni, Pessach. On a generalization of the linear homogeneous function. Riveon Lematematika **8**, 16–29 (1954). (Hebrew. English summary)

The author considers a not necessarily commutative ring R with identity where an absolute value is defined satisfying the conditions: $|0|=0$; $|-a|=|a|>0$, $a\neq 0$; $|a+b|\leq|a|+|b|$; $|ab|\leq A|a|\cdot|b|$, $A>0$, A independent of a and b. He further assumes that R is complete in respect to this valuation. Integration and differentiation are introduced as abstract operations satisfying the conditions: every $a \in R$ has a unique integral $\int a \in R$, and the derivative a' is unique whenever it exists; the rules of derivation of sums and products are the usual ones, so that the set of all elements with derivatives is a subring R_d of R; the set of all integrals is a two-sided ideal of R_d; finally, an assumption is made concerning the absolute values of certain multiple integrals. It follows that if $u \in R_d$, then u has a unique representation of the form $u=\int v+c$, where c is a constant (i.e., $c'=0$) which is called the initial value of U and denoted by $c=A_u(u)$. Of importance are the so called perfect elements, i.e., elements a with an inverse a^{-1}, such that $a, a^{-1} \in R_d$. A function $f(y)$ defined in R_d is called an L-function (linear) if the following relations hold: $f(y\pm z)=f(y)\pm f(z)$; $xf(y)=f(xy)$; $\int f(y)=f(\int y)$; if $k'=0$, then $(f(k))'=0$. In a previous paper [Compositio Math. **5**, 403–429 (1938)] the author has shown how the study of the solutions of certain linear integro-differential equations may be reduced to the study of linear differential equations in R. In the present note he studies the relationship between the following two equations in R: 1) $xy''+f(y')+xg(y)=0$; 2) $xz''+f(z')+2z'+xg(z)=0$. Here $f(y)$ and $g(y)$ are assumed to be L-functions. His results are: I) If y is a solution of 1) and if y''' exists, then $z=x^{-1}y'$ is a solution of 2). II) If z is a solution of 2) and if the equation $g(c)=-A_u(xz'+f(z)+z)$ has a constant solution c, then $y=\int(xz)+c$ is a solution of 1). III) Let a and b be constant and perfect, $c=A_u(x)$, $ac=ca$, z a solution of the equation $xz''+f(z')+2z'+xazb=0$. Then

$$y=\int(xz)-a^{-1}A_u(xz'+f(z)+z)b^{-1}$$

is a solution of the equation $xy''+f(y')+xayb=0$.
J. Levitzki (Jerusalem).

Nobusawa, Nobuo. An extension of Krull's Galois theory to division rings. Osaka Math. J. **7**, 1–6 (1955).

La première partie de ce travail est un exposé de la théorie de Galois pour une extension de rang fini d'un corps non commutatif, qui ne semble pas différer sensiblement des exposés connus de Cartan, Jacobson et Nakayama. S'appuyant sur ces résultats, l'auteur montre dans la seconde partie comment on peut les généraliser à certains types d'extension de rang infini en topologisant le groupe de Galois à la manière de Krull, mais il doit faire sur l'extension considérée l'hypothèse qu'elle est localement finie (i.e., l'extension engendrée par un nombre fini quelconque d'éléments est de rang fini) et que l'"orbite" d'un élément (ensemble de ses transformés par le groupe de tous les automorphismes sur le corps de base) est toujours fini. *J. Dieudonné.*

Herz, Jean-Claude. Contribution à la théorie algébrique des équations aux dérivées partielles. Ann. Sci. Ecole Norm. Sup. (3) **71**, 321–362 (1954).

Let K be a field of functions meromorphic in a given region in the space of n complex variables $x_1, \cdots, x_n$ and let K be closed with respect to the n partial derivations $\partial/\partial x_i$. These derivations generate a vector space E over K of dimension n, and the mapping $(X, Y)\rightarrow\{X, Y\}=XY-YX$ defines an internal law of composition on E with respect to which E is almost (but in general not quite) a Lie algebra over K, the defect being that in general $\{\alpha X, Y\}\neq\alpha\{X, Y\}$. Abstracting the properties of $\{\ ,\ \}$ the author defines the notion of a pseudo Lie algebra E over a (not necessarily commutative) field K. He gives a characterization of all pseudo Lie algebras of dimension >1 (the cases of commutative and noncommutative K requiring separate treat-

ment); results are less complete for dimension 1, being limited to more special situations. He shows, following G. Birkhoff on Lie algebras [Ann. of Math. (2) **38**, 526–532 (1937)], that if K is commutative and of characteristic $\neq 2$ and dim $E>1$ then E can be embedded in a vector space with associative multiplication $*$ (the enveloping pseudo algebra) such that $\{u, v\}=u*v-v*u$ $(u, v \in E)$. The author then reverts to the case in which K and E are as in the beginning (or the generalization in which K is an abstract differential field). Every homogeneous linear partial differential equation in one unknown Y can be written in the form $wY=0$ with w in E; for every system S of such equations the system of operators w generates a pseudo Lie subalgebra $(S)_E$ of E. The main theorem asserts that two systems S, S' have precisely the same solutions if and only if $(S)_E=(S')_E$; the proof employs J. F. Ritt's theorem of zeros for differential polynomials. When K and E are as in the beginning, the author also shows how the general solution of S is an arbitrary function of $n-g$ independent functions, where $q=\dim (S)_E$. *E. R. Kolchin.*

Leptin, Horst. Linear kompakte Moduln und Ringe. Math. Z. **62**, 241–267 (1955).

A linearly compact Λ-module is a Λ-module with submodule neighborhoods of zero (i.e., linearly topologized) and such that every collection of cosets of closed submodules has nonvoid intersection provided it has the finite intersection property. A topological ring R is linearly compact if the left R-module R is. The principal results of Part I (Linearly compact modules) are: (1) The topology of a linearly compact module can be coarsened (weakened) to a unique coarsest topology in which the module is still linearly compact. The two topologies then have the same closed submodules. [Reviewer's comment: this shows that a module linearly compact in the discrete topology has a very natural topology (viz., the unique coarsest linearly compact one); e.g., this exhibits the naturalness of the local topology in complete local rings and explains some counterexamples of the reviewer [Amer. J. Math. **75**, 79–90 (1953); MR **14**, 532].] (2) A still stronger condition on a linearly compact topology than being a coarsest one is linear compactness "im engeren Sinne": every continuous homomorphism into a linearly topologized module is open. The author gives several equivalent conditions, one of which is essentially the one originally used by van Dantzig: minimum condition modulo open submodules.

Part II (Linearly compact rings) comprises some results on transfinite nilpotence of the radical, especially for rings linearly compact "im engeren Sinne", and the analog of the Wedderburn-Artin theory (providing the proper generalization of results of the reviewer, who assumed ideal instead of left-ideal-neighborhoods of zero): A linearly compact ring is semisimple if and only if it is the complete direct sum of completely primary rings. A completely primary ring (all linear transformations on a vector space over a division ring) is the only possible nonradical linearly compact ring with no proper closed ideals. Furthermore, the completion of any semisimple ring in its natural topology (maximal left ideals as subbase at zero) is linearly compact and semisimple so that Jacobson's structure theory follows from the present theorems.

Part III (Hyperdirect decompositions) on the possibility of special direct decompositions of linearly topologized modules results in the following result among others: A linearly compact ring with unit is uniquely decomposable as a complete direct sum of closed two-sided ideals which are not further decomposable. *D. Zelinsky* (Kyoto).

Theory of Groups

Tamura, Takayuki. On a monoid whose submonoids form a chain. J. Gakugei. Tokushima Univ. Math. **5**, 8–16 (1954).

By a monoid the author means a set M endowed with an associative binary operation; the more usual term "semigroup" will be used in this review. M is called a Γ-semigroup if the set of its subsemigroups is totally ordered by inclusion. It was shown by R. Baer [Amer. J. Math. **61**, 1–44 (1939)] that a finite group is a Γ-semigroup if and only if it is cyclic of prime power order. In the present paper, all Γ-semigroups are found. There is, to within isomorphism, exactly one infinite Γ-semigroup for each prime p, namely the additive group of all natural numbers m/p^n $(m=0, 1, 2, \cdots, p^{n-1};$ $n=0, 1, 2, \cdots)$ taken modulo 1. The finite Γ-semigroups are just those cyclic semigroups $\{a, a^2, \cdots\}$ the maximal subgroup K of which has order a power of a prime p, and such that one of the following conditions holds: (1) $a \in K$; (2) $a^2 \in K$; (3) $a^3 \in K$ and $p\neq 2$. *A. H. Clifford.*

Tamura, Takayuki. Notes on finite semigroups and determination of semigroups of order 4. J. Gakugei. Tokushima Univ. Math. **5**, 17–27 (1954).

In this article, a complete determination is made of all semigroups of order 4. They are classified as follows: I. Unipotent, i.e., having exactly one idempotent. II–V. Four different types of non-commutative semigroups classified according to type of greatest commutative homomorphic image (analogous to group modulo commutator subgroup). VI. Commutative semigroups not in I. There are 126 semigroups in the list, 58 of them commutative, and with anti-isomorphs omitted. This is an agreement with Forsythe [see the following review]. *A. H. Clifford.*

Forsythe, George E. SWAC computes 126 distinct semigroups of order 4. Proc. Amer. Math. Soc. **6**, 443–447 (1955).

This paper reports the determination of all semigroups of order 4 on the electronic computer SWAC. Two semigroups are regarded as distinct if they are neither isomorphic nor antiisomorphic. In this sense, it is found that there are 126 distinct semigroups of order 4, of which 58 are commutative, in agreement with Tamura [see the preceding review].

A. H. Clifford (New Orleans, La.).

Schwarz, Štefan. The theory of characters of finite commutative semigroups. Czechoslovak Math. J. **4(79)**, 219–247 (1954). (Russian. English summary)

By a character of a semigroup S is meant a complex-valued function χ on S such that $\chi(ab)=\chi(a)\chi(b)$ for all $a,b \in S$. The main results of the paper hold only when S is finite and commutative, which we henceforth assume. If we define the product of two characters of S in the natural way, the set of all characters of S becomes a semigroup S^*. S^* is a finite commutative semigroup with zero element χ_0. The main objective of the paper is to give a complete description of the structure of S^* assuming that of S is known.

Let E be the semi-lattice of idempotent elements of S. Let the elements of E be labelled by an index set I. To each $e_i \in E$ $(i \in I)$ corresponds a subsemigroup P_i of S consisting

of all $a \in S$ such that $a^n = e_i$ for some n. These so-called maximal subsemigroups P_i are disjoint, and their union is S. The maximal subgroup G_i of S having e_i as identity element is the group-ideal, or Suschkewitsch kernel, of P_i. An ideal J of S is called prime if $S-J$ is a subsemigroup of S; S itself and the empty set ϕ are also considered to be prime ideals of S. There is an order-isomorphism between E and the set $\mathfrak{P}'$ of all prime ideals $J \neq S$ of S whereby, if $e_i \leftrightarrow J_i$, e_i is the least member of E not in J_i, and J_i is the union of all those P_j for which e_j is not $\geq e_i$. There is an order-anti-isomorphism between the (semi-)lattice E^* of all idempotent elements ϵ of S^* and the set $\mathfrak{P} = \mathfrak{P}' \cup \{S\}$ of all prime ideals of S whereby, if $\epsilon_i \leftrightarrow J_i$, ϵ_i is the characteristic function of $S-J_i$, and the zero character χ_0 corresponds to the prime ideal S.

If $\chi \in S^*$, the set of all $a \in S$ such that $\chi(a)=0$ is a prime ideal J of S. If $\chi \neq \chi_0$ then $J \neq S$, and $J = J_i$ for some $i \in I$. If b is any element of S we have $\chi(b) = \chi(be_i)$ if $be_i \in G_i$ and $\chi(b)=0$ otherwise. χ is thus uniquely determined by the character of G_i it induces. Conversely, if ψ is any character of G_i, χ defined by $\chi(b) = \psi(be_i)$ if $be_i \in G_i$ and $\chi(b)=0$ otherwise is a character of S. Hence the set of all $\chi \in S^*$ vanishing just on J_i is a subgroup G_i^* of S^* isomorphic with the character group of G_i, and hence with G_i itself; ϵ_i is the identity element of G_i; and S^* is the union of the disjoint groups G_i^* $(i \in I)$ and $\{\chi_0\}$. The "gross structure" of S^* is thereby determined: S^* is a known (semi-)lattice E^* $(\cong\mathfrak{P})$ of known groups G_i^* $(\cong G_i)$ and $\{\chi_0\}$. In order to obtain the "fine structure" of S^*, the author shows that $G_j^*\epsilon_i \cong G_i e_j$ for every pair of indices $i,j \in I$ such that $e_j < e_i$ (and hence $\epsilon_i < \epsilon_j$). [The reviewer feels that the fine structure of S is more convincingly established by the observation that the homomorphism $\chi_j \to \chi_j\epsilon_i$ of G_j^* into G_i^* is just the adjoint of the homomorphism $a_i \to a_i e_j$ of G_i into G_j.] It is noteworthy that S^* is determined by the "group part" $S_0 = \bigcup_{i \in I} G_i$ of S. *A. H. Clifford* (New Orleans, La.).

Schwarz, Štefan. Characters of commutative semigroups as class functions. Czechoslovak Math. J. 4(79), 291–295 (1954). (Russian. English summary)

This paper continues that of the preceding review. Two elements a, b of P_i are called conjugate if $ae_i = be_i$. Any character χ of S assumes the same value for conjugate elements; and if a is not conjugate to b, there exists a character χ of S such that $\chi(a) \neq \chi(b)$. *A. H. Clifford.*

Schwarz, Štefan. On a Galois connexion in the theory of characters of commutative semigroups. Czechoslovak Math. J. 4(79), 296–313 (1954). (Russian. English summary)

Let S be a finite commutative semigroup, and S^* the semigroup of characters of S [see the two preceding reviews]. An ideal $\mathfrak{a}$ of S is called closed if $x^n \in \mathfrak{a}$ $(x \in S$, n a positive integer) implies $x \in \mathfrak{a}$. All ideals of S^* are closed. If $\mathfrak{a}$ is an ideal of S, the set $\mathfrak{a}_0$ of all characters of S vanishing on $\mathfrak{a}$ is an ideal of S^*. If $\mathfrak{a}_0$ is an ideal of S^*, the set $\mathfrak{a}$ of all elements $a \in S$ such that $\chi(a)=0$ for every $\chi \in \mathfrak{a}_0$ is an ideal of S. It is shown that these two mappings define a one-to-one inclusion reversing correspondence between the set of all closed ideals of S and the set of all ideals of S^*. If $\mathfrak{a}$ is any ideal of S, the semigroup $[\mathfrak{a}]^*$ of all characters of the semigroup $\mathfrak{a}$ is isomorphic to the Rees factor semigroup $S^*/\mathfrak{a}_0$. The semigroup $[S/\mathfrak{a}]^*$ of all characters of $S/\mathfrak{a}$, with the unit character removed, is isomorphic with $\mathfrak{a}_0$.

A. H. Clifford.

★Burnside, W. Theory of groups of finite order. 2d ed. Dover Publications, Inc., New York, 1955. xxiv+512 pp. Clothbound: \$3.95; paperbound: \$2.00.

Reprint by photo-offset of the 2d edition [Cambridge, 1911].

Kertész, Andor. Abelian torsion groups. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 3, 111–126 (1954). (Hungarian)

The results of this paper have been previously published in Acta Math. Acad. Sci. Hungar. 3, 121–126, 225–232 (1952); Acta Sci. Math. Szeged 15, 61–69 (1953) [MR 14, 617, 945; 15, 99].

Fuchs, L., Kertész, A., and Szele, T. On abelian groups whose subgroups are endomorphic images. Acta Sci. Math. Szeged 16, 77–88 (1955).

The last two authors studied abelian groups for which every finitely generated subgroup is an endomorphic image [same Acta 15, 70–76 (1953); MR 15, 196]. Sąsiada [Bull. Acad. Polon. Sci. Cl. III. 2, 359–362 (1954); MR 16, 565] made the stronger assumption that every countable subgroup is an image. In this third paper on the subject the authors characterize abelian groups for which every subgroup is an endomorphic image. In the case (probably most interesting) of a primary group the condition turns out to be the equality of two cardinal numbers. Define the final rank of G to be the minimum of the ranks of p^nG. Then the desired condition is the equality of the final rank of G and of a basic subgroup of G (the second of these cardinal numbers can also be described as the lim sup of the first $\aleph_0$ Ulm invariants). Suitable additional considerations give explicit results in the torsion-free and mixed cases.

I. Kaplansky (Chicago, Ill.).

Kertész, Andor. Algebraically closed and free groups. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 229–236 (1954). (Hungarian)

Hungarian version of Publ. Math. Debrecen 3, 174–179 (1953); MR 15, 775.

Sato, Shoji. On the lattice homomorphisms of infinite groups. II. Osaka Math. J. 6, 109–118 (1954).

[For part I see same J. 4, 229–234 (1952); MR 14, 618.] A lattice homomorphism of a group onto a lattice is called regular if it maps the lower kernel onto the least element. The author studies groups admitting regular lattice homomorphisms onto a direct product of r two-element chains, proving that a solvable group G has this property if and only if (1) G is a torsion group containing a normal subgroup N and a subgroup H such that $G=NH$, $N \cap H = 1$, where H is a direct product of p_i-groups P_i $(i=1, 2, \cdots, r)$, $p_i \neq p_j$ for $i \neq j$, each P_i being a locally cyclic group or a generalized quaternion group having intersection $\neq 1$ with the center of G, and N contains no element of order p_i, or (2) G contains elements of infinite order, and (a) the totality of elements of finite order is a subgroup M such that G/M is locally cyclic, (b) G has a normal subgroup N, $N \geq M$, such that G/N is locally cyclic and a direct product of locally cyclic p_i-groups P_i/N $(i=1, 2, \cdots, r)$, (c) for $a \in P_i$, a non-$\in N$, $n \in N$, the intersection of the cyclic group generated by a with the centralizer of n is not part of N, and (d) M contains no element of order p_i. [For a discussion of this and related problems in the case of finite groups see M. Suzuki, Trans. Amer. Math. Soc. 70, 372–386 (1951); MR 12, 587.] *D. G. Higman* (Missoula, Mont.).

Ono, Takashi. Arithmetic of orthogonal groups. J. Math. Soc. Japan 7, 79–91 (1955).

Let K be a field of characteristic $\neq 2$ and let V be an n-dimensional vector space over K. The group of automorphisms of V over K is denoted by $GL(V)$. Let f be a non-degenerate, symmetric bilinear form on V. For any $\sigma \in GL(V)$ let $f^\sigma(x, y) = f(\sigma x, \sigma y)$, $x, y \in V$. Two forms f, g are said to be congruent, written $f \sim g$, if $g = f^\sigma$ for some $\sigma \in GL(V)$ and are said to be similar in K, written $f \approx g$, if $g \approx af$ for some non-zero $a \in K$. The set of all $\sigma \in GL(V)$ such that $f^\sigma = f$ is called the orthogonal group corresponding to f and is denoted by $O(V, f)$. If f, g are two forms on V, then $O(V, f)$ and $O(V, g)$ are conjugate in $GL(V)$ if, and only if, $f \approx g$ in K. Now let K be either a field of algebraic numbers or a field of algebraic functions of one variable over a finite field of characteristic $\neq 2$. Let K_p be a p-adic completion of K with respect to a place p in K. Denote by V_p the scalar extension of V with respect to K_p. If f is a form on V, then f may be considered as a form on V_p and $O(V, f)$ may be regarded as a subgroup of $O(V_p, f)$. The main results of the paper are contained in the following theorems which are examples of what the author calls the "Hasse principle." 1. $f \approx g$ in K if and only if $f \approx g$ in K_p for every place p in K. 2. $O(V, f)$ and $O(V, g)$ are conjugate in $GL(V)$ if and only if $O(V_p, f)$ and $O(V_p, g)$ are conjugate in $GL(V_p)$ for every place p in K. 3. If $O(V_p, f)$ and $O(V_p, g)$ are isomorphic for every place p in K, then $O(V, f)$ and $O(V, g)$ are conjugate in $GL(V)$. *C. E. Rickart.*

Moriya, Mikao. Theorie der 2-Cohomologiegruppen in diskret bewerteten perfekten Körpern. Proc. Japan Acad. 30, 787–790 (1954).

Let $k \subset K_1 \subset K$ be a tower of finite separable algebraic extensions of a field k which is complete for a discrete valuation. Let r, R_1, R denote the rings of integers for k, K_1, K, respectively. Let $H^2(R/r; R)$ stand for the 2-dimensional cohomology group computed from the normalized symmetric r-linear cochains for R in R (and interpret the other groups $H^2(\cdot/\cdot; \cdot)$ analogously). A number of results concerning these groups are stated without proof, of which the essential ones are as follows. (1) The canonical R-homomorphism sequence

$$(0) \to H^2(R/R_1; R) \to H^2(R/r; R) \to H^2(R_1/r; R) \to (0)$$

is exact. (2) $H^2(R/r; R)$ is a finitely generated R-module which is annihilated by the different $D(K/k)$ of K/k; and $D(K/k)$ is the nth power of the valuation ideal of R, where n is the length of a composition series for the R-module $H^2(R/r; R)$. *G. P. Hochschild* (Berkeley, Calif.).

Brauer, Richard, and Tate, John. On the characters of finite groups. Ann. of Math. (2) 62, 1–7 (1955).

If χ_i is an irreducible character of a finite group G then any linear combination of the χ_i with coefficients in the ring of rational numbers Z is called a 'generalized character' of G and belongs to the character ring $X = X_Z(G)$ of G. If ψ_i is an irreducible character of an "elementary sub-group" G_i of G and ψ_i^* the corresponding induced character of G, then Brauer has proved [Ann. of Math. (2) 48, 502–514 (1947); MR 8, 503] that (A) every member χ of X can be written in the form $\chi = \sum a_i \psi_i^*$ with coefficients in Z. In a subsequent paper [ibid. (2) 57, 357–377 (1953); MR 14, 844] he showed that this result is equivalent to the following condition: (B) a complex-valued function θ defined in a finite group G belongs to $X_Z(G)$ if and only if (1) θ is a class function, (2) for every sub-group G_i the restriction $\theta | G_i$ of θ to G_i belongs to $X_Z(G_i)$ of G_i. Of course, the basis of these ideas is Frobenius' reciprocity theorem and the goal of this paper is to clarify the connection between (A) and (B). Let R be an integral domain containing Z. If $\{H_\alpha\}$ is a fixed family of subgroups of G let $\{\psi_{\alpha j}\}$ denote the set of irreducible characters of H_α. Define an R-module $V_R = \sum_{\alpha, j} R\psi^*_{\alpha j}$, and an R-module U_R made up of functions θ satisfying the conditions of (B) for the set of sub-groups $\{H_\alpha\}$. Theorem 1 states that U_R is a ring, V_R is an ideal of U_R and $U_R \supseteq X_R \supseteq V_R$. In the corrollary of Theorem 3 the authors conclude that if every elementary subgroup G_i is contained in some subgroup H_α then $U_Z = X_Z(G) = V_Z$, proving the equivalence and the correctness of (A) and (B) in an elegant manner. *G. de B. Robinson* (Toronto, Ont.).

Braconnier, Jean. Les algèbres de groupes et leurs représentations. Ann. Univ. Lyon. Sect. A. (3) 15, 27–34 (1952).

An expository article, presumably first in a series, laying the groundwork for a survey of the present state of the theory, especially as developed by Weil, Gelfand and Raikov, Segal, Godement, Gelfand and Neumark, and Mautner. *J. G. Wendel* (Ann Arbor, Mich.).

Monna, A. F. Sur une propriété du groupe topologique additif des nombres réels. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17, 295–300 (1955).

Let R be the additive group of real numbers. The theorem proved asserts that a topological group G in which some neighborhood of the identity is homeomorphic to an open interval in R is locally isomorphic to R. The usual proof is here replaced by a proof which uses Haar measure in G. *P. A. Smith* (New York, N. Y.).

***Tits, J. Généralisation d'un théorème de Kerekjarto.** IIIe Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 64–65. Fédération belge des Sociétés Scientifiques, Bruxelles.

The purpose of the investigation sketched in this communication is to characterize all n-tuply transitive (i.e., simply transitive on the ordered n-tuples) continuous groups of topological transformations of a variety in itself, for $n \geqq 3$. From results obtained in the author's thesis [Bruxelles, 1950] none such exist for $n > 3$. Concepts and theorems due to Freudenthal and Pontrjagin lead to the conclusion that for $n = 3$ the group is homeomorphic to the group of homographies over the real or the complex line. *L. M. Blumenthal* (Columbia, Mo.).

Mostow, G. D. Some new decomposition theorems for semi-simple groups. Mem. Amer. Math. Soc. no. 14, 31–54 (1955).

Let L' be a Lie algebra over the complex field and let L'' denote the same algebra L' when considered as a Lie algebra over the real field. A subalgebra L^* of L'' is called a compact real form of L' if L'' is the direct sum of L^* and $\sqrt{(-1)}L^*$ and if the analytic subgroup of the adjoint group of L'' corresponding to L^* is compact. Let L be a real semi-simple Lie algebra and L' the complexification of L. Given any compact real form L^* of L' which is invariant under complex-conjugation, L is then decomposed into the direct sum $L = C + D$, where $C = L \cap L^*$, $D = L \cap \sqrt{(-1)}L^*$. Such a decomposition is called a Cartan decomposition of L.

The main results of the paper are then stated as follows: 1) Let G be a connected semi-simple Lie group and let $L = C + D$ be a Cartan decomposition of its Lie algebra L

as given above. Let D_1 be a linear subspace of D such that $[X[X, Y]] \in D_1$ for all $X, Y \in D_1$ and let D_2 be the orthogonal complement of D_1 in D with respect to the fundamental bilinear form on L. Then G decomposes topologically into the product $G = KE_1E_2$ where K is the analytic subgroup of G determined by C, $E_1 = \exp D_1$, $E_2 = \exp D_2$. 2) Let L be a real semi-simple Lie algebra and S a semi-simple subalgebra of L. For any Cartan decomposition $S = C_S + D_S$ of S, there exists a Cartan decomposition $L = C + D$ of L such that $C_S \subset C$, $D_S \subset D$. 3) Let

$$L_1 \subset L_2 \subset \cdots \subset L_n$$

be a sequence of linear semi-simple Lie algebras acting on a real linear space V of finite dimension. Then a positive definite inner product B for V can be defined in such a way that $L_i = L_i \cap C + L_i \cap D$ where C and D are respectively the sets of skew-symmetric and symmetric linear transformations of V with respect to B.

It follows from these results, the author says, that the factor space of a Lie group by a semi-simple subgroup is a fiber bundle whose fibers are euclidean and whose base space is the factor space of a compact group.

For the proof of 1), 2), 3), the author studies the properties of the symmetric Riemannian space formed by all $n \times n$ positive definite symmetric matrices, thus giving, with simplicity and rigor, a new exposition of Cartan's theory of symmetric Riemannian spaces, including a proof of the Cartan's conjugacy theorem on maximal compact subgroups of a connected semi-simple Lie group. *K. Iwasawa.*

Mostow, G. D. Self-adjoint groups. Ann. of Math. (2) 62, 44–55 (1955).

The main theorem of the paper is stated as follows: Let G be an algebraic group of linear transformations on a real or complex linear space V. If G is fully reducible, then a positive definite hermitian form may be introduced in V with respect to which every linear transformation of G is self-adjoint.

An outline of the proof is as follows: Let G be a real linear Lie group acting on a real linear space V and L the corresponding linear Lie algebra of G. Let L' denote the complexification of L and G' the corresponding complex linear analytic group acting on the complexification ${}^cV'$ of V. The group $G' = GG_0'$ is then called the complexification of the Lie group G. A compact subgroup K of G' is called an invariant compact real form of G (and G') if $G' = KG_0'$, K is invariant under complex conjugation and if L' is, as a real linear space, the direct sum of L_K and $\sqrt{(-1)}L_K$, L_K being the Lie algebra of K. Now, it is proved, using the theory of Lie groups extensively, that if a real (or complex) linear group G (or G') acting on a linear space V (or V') has an invariant compact form K as defined above, then a positive definite hermitian form can be introduced on V (or V') with respect to which every transformation of G (or G') is self-adjoint.

On the other hand, applying theorems of Chevalley and Jacobson, it is also proved that a fully reducible real (or complex) linear algebraic Lie group always has an invariant compact form. The main theorem then follows immediately from this and the result mentioned above.

In carrying out the proof, the following result is also obtained [see the paper reviewed above]: Let $G_1 \subset G_2 \subset \cdots \subset G_n$ be a sequence of algebraic Lie groups acting on a real or complex vector space. If there exists, for each i, a positive definite hermitian form with respect to which G_i is self-adjoint, then there exists also a positive definite hermitian form with respect to which $G_1, \cdots, G_n$ are simultaneously self-adjoint. *K. Iwasawa* (Cambridge, Mass.).

NUMBER THEORY

Barsotti, Leo. Some theorems on numerical divisibility. Soc. Parana. Mat. Annuário 1, 14–17 (1954). (Portuguese)

The following theorem of B. Ram [J. Indian Math. Club 1, 39–43 (1909)] is generalized to multinomial coefficients: If $a > 1$ is the greatest common divisors of the binomial coefficients $\binom{n}{k}$ $(k = 1, 2, \cdots, n-1)$, then n is a power of a prime p and $a = p$. An unnecessarily complicated proof is given of the fact that

$$(2n)!! \equiv (-1)^n (2n-1)!! \pmod{2n+1}$$

which follows at once by reversing the order of the factors. *D. H. Lehmer* (Berkeley, Calif.).

Sagastume Berra, A. E. Schönberg's problem. Univ. Nac. Tucumán. Rev. Ser. A. 10, 7–17 (1954). (Spanish)

The composer Arnold Schönberg in his study of musical scales was led to the following problem: To find a permutation $(a_1, a_2, \cdots, a_n)$ of the n integers $0, 1, 2, \cdots, n-1$, such that the differences $a_{i+1} - a_i$ are all incongruent modulo n. There is no loss in generality if one assumes that $a_1 = 0$. For the problem to be possible it is necessary that n be even and $a_n = n/2$. Conversely, if n is even, the permutation

$$(0, 1, n-1, 2, n-2, \cdots)$$

is a solution of the problem. The author considers various transformations which take solutions into solutions, such as the multiplication of each a_i by a constant prime to n. The problem of enumeration of solutions for a given n is discussed briefly. For $n = 4$ there are but two solutions (0132), (0312). For $n = 6, 8, 10$ there are 4, 24, 288 solutions respectively. For $n = 12$, the case considered by Schönberg, the number is very large but unknown. *D. H. Lehmer.*

Selmer, Ernst S. The indeterminate equation $X^3 + Y^3 = AZ^3$. Nordisk Mat. Tidskr. 3, 48–56, 80 (1955). (Norwegian. English summary)

The author gives an account of some results from an earlier paper [Acta Math. 85, 203–362 (1951); MR 13, 13]. *W. Ljunggren* (Bergen).

Rosenthall, E. Reducible diophantine equations and their parametric representations. Canad. J. Math. 7, 328–336 (1955).

The diophantine equation

$$(*) \qquad \prod (\mathrm{Norm}_{\mathfrak{K}_i/\mathfrak{R}}\, \kappa_i)^{c_i} = 1$$

is to be solved in $\kappa_i \in \mathfrak{K}_i$, where the $\mathfrak{K}_i$ $(1 \leq i \leq I)$ are given algebraic fields over the rational field $\mathfrak{R}$ and the c_i are given nonzero integers in $\mathfrak{R}$. The author shows that the linear factors into which the left side of (*) splits in the least normal extension $\mathfrak{N}/\mathfrak{R}$ containing the $\mathfrak{R}_i$ must have ideal factorisations of one of a finite number of parametric shapes;

but he does not consider when these ideal factorisations correspond to solutions κ_i of (*). *J. W. S. Cassels.*

***Gloden, A.** **Théorèmes nouveaux sur les systèmes multigrades d'ordre n et applications.** III° Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 80–84. Fédération belge des Sociétés Scientifiques, Bruxelles.

Various elementary theorems are given concerning the Tarry-Escott problem; for definitions see Gloden's book on the subject or the review thereof [Mehrgradige Gleichungen, Noordhoff, Groningen, 1944; MR 8, 441]. The applications are to geometry, such as the determination of a quadrilateral inscribed in a circle with rational values for sides, diagonals, and area. *I. Niven* (Berkeley, Calif.).

Dénes, P., and Turán, P. **A second note on Fermat's conjecture.** Publ. Math. Debrecen 4, 28–32 (1955).

As in a previous paper [Turán, J. Indian Math. Soc. (N.S.) 15, 47–50 (1951); MR 13, 321], the authors denote by $R_q(N)$ the number of solutions (x, y, z) of the system

$$x^q+y^q-z^q=0,\quad 1\leqq x,\ y,\ z\leqq N,$$

where q is an odd prime and $qh=1$. An improved inequality

$$R_q(N)<q(1+3\cdot 2^h)N^{2h}$$

is established by an elementary argument about the prime factors of $(z^q-x^q)(z-x)$. By using a deeper theorem of Furtwängler and some results on Dirichlet L-functions the author proves that

$$R_q(N)<cN^{2h}(\log N)^{2(h-1)}.$$

He conjectures that the state of the art permits a proof of $R_q(N)<cN^h$, but feels that $R_q(N)=O(N^\epsilon)$ $(\epsilon>0)$, is very deep. *D. H. Lehmer* (Berkeley, Calif.).

Ricci, Giovanni. **Funzioni aritmetiche e quasi-asintoticità.** Rend. Sem. Mat. Fis. Milano 24 (1952–53), 88–106 (1954).

In this paper, which is partly expository in nature, the author reviews various problems concerned with the order of magnitude of arithmetical functions. The notions of asymptoticity in the mean, quasi-asymptoticity and quasi-asymptoticity in the mean are defined and discussed. All of these concepts are copiously illustrated with examples taken from the classical literature. *A. L. Whiteman.*

Ricci, Giovanni. **Sul pennello di quasi-asintoticità della differenza di interi primi consecutivi. I, II.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 17, 192–196 (1954); 17 (1954), 347–351 (1955).

Let $b_n>0$ for all subscripts n with the possible exception of the values n_k of a sequence $\{n_k\}$ of asymptotic density zero. Then a_n is said to be quasi-asymptotic to b_n as $n\to\infty$ (written $q\lim a_n/b_n=1$) if for every $\epsilon>0$ the sequence $n(\epsilon, h)$ $(h=1, 2, \cdots)$ of the indices n for which

$$(1-\epsilon)b_n<a_n<(1+\epsilon)b_n$$

has asymptotic density 1. The sequence $\{a_n, b_n\}$ determines a quasi-asymptotic pencil of the sequence $\{x_n\}$ if $q\liminf x_n/a_n=1$ and $q\limsup x_n/b_n=1$. Theorem I. If p_n denotes the nth prime, then $p_{n+1}-p_n$ is not quasi-asymptotic to any monotonic function $\psi(n)$. Theorem II. Let $\{\alpha\log p_n, \beta\log p_n\}$ be a quasi-asymptotic pencil of the sequence $p_{n+1}-p_n$. Then the uniquely determined constants α and β satisfy the inequalities $\alpha\leqq 1-H/C$, $\beta\geqq\alpha+2H/C$, where $H=\prod(1-(p-1)^{-2})$ (the product extending over odd primes) and $0\leqq C\leqq 16H$. Theorem III. Let $\psi_1(n)$ and $\psi_2(n)$ be two monotonic functions forming a quasi-asymptotic pencil of the sequence $p_{n+1}-p_n$. Then $\psi_2(n)\to\infty$ and

$$\liminf \psi_1(n)/\log p_n\leqq 1-H/C,$$
$$\limsup (\psi_2(n)-\psi_1(n))/\log p_n\geqq 2H/C.$$

A. L. Whiteman (Los Angeles, Calif.).

Vinogradov, I. M. **Part I. Improvement of the remainder term of some asymptotic formulas. Part II. An upper bound of the modulus of a trigonometric sum. Part III. General theorems on the upper bound of the modulus of a trigonometric sum.** Amer. Math. Soc. Translation no. 94, 66 pp. (1953).

Translated from Izv. Akad. Nauk SSSR 13, 97–110 (1949); 14, 199–214 (1950); 15, 109–130 (1951); MR 11, 233; 12, 161; 13, 328.

***Arzt, Sholom.** **On a mean value theorem for certain divisor functions taken over exponential sequences.** Abridgment of a dissertation, New York University, 1951. 5 pp.

The author summarizes some results obtained in his doctoral dissertation concerning the asymptotic evaluation of sums of the type $\sum_{n\leqq x}\sigma_{-s}(f(n))$, where $f(n)=g(b_1^n, b_2^n)$, with $g(x, y)$ a homogeneous polynomial and b_1, b_2 positive or negative integers. Previous results of this nature were obtained by the reviewer [Duke Math. J. 17, 159–168 (1950); MR 11, 715] for the case where $f(n)$ is a polynomial in n. *R. Bellman* (Santa Monica, Calif.).

Carlitz, L. **The number of solutions of certain types of equations in a finite field.** Pacific J. Math. 5, 177–181 (1955).

Let $f_i(x_i)=f_i(x_{i1}, \cdots, x_{is_i})$ $(i=1, \cdots, r)$ denote r polynomials with coefficients in GF(q) and assume

$$f_i(\lambda x_i)=\lambda^{m_i}f_i(x_i)\quad (\lambda\in \mathrm{GF}(q));$$

assume also $(m, m_i, q-1)=1$ $(i=1, \cdots, r)$. The main result of the author states that the total number N of solutions of

$$y^m=f_1(x_{11}, \cdots, x_{1s_1})+\cdots+f_r(x_{r1}, \cdots, x_{rs_r})$$

in $s_1+\cdots+s_r+1$ unknowns is given by $N=q^{s_1+\cdots+s_r}$. The method employed is an application of a simple principle due to M. Ward [Proc. Nat. Acad. Sci. U. S. A. 37, 113–114 (1951); MR 13, 13]. *A. L. Whiteman.*

Košlyakov, N. S. **Errata: Investigation of some questions of the analytic theory of a rational and quadratic field. I.** Izv. Akad. Nauk SSSR. Ser. Mat. 19, 271 (1955). (Russian)

See same Izv. 18, 113–144 (1954); MR 16, 15.

Földes, István. **Sowjetische Ergebnisse in der Theorie der algebraischen Zahlkörper.** Mat. Lapok 3, 179–202 (1952). (Hungarian. Russian and German summaries)

Bambah, R. P., and Rogers, K. **An inhomogeneous minimum for nonconvex star-regions with hexagonal symmetry.** Canad. J. Math. 7, 337–346 (1955).

For certain functions $f(x, y)$ it is well known that there exist real numbers x, y taking assigned residues mod 1 and satisfying an inequality of the type

$$|f(x, y)|\leqq\max\{|f(\tfrac{1}{2}, 0)|, |f(0, \tfrac{1}{2})|, |f(\tfrac{1}{2}, \tfrac{1}{2})|, |f(\tfrac{1}{2}, -\tfrac{1}{2})|\}.$$

L. J. Mordell [Duke Math. J. 19, 519–527 (1952); MR 14, 540] investigated the conditions on such a function when the region $|f(x, y)|\leqq 1$ has one asymptote, while K. Rogers [J. London Math. Soc. 28, 394–402 (1953); MR 15, 106] considered the case of two asymptotes. Here the authors

generalize the work of R. P. Bambah [Proc. Cambridge Philos. Soc. 47, 457–460 (1951); MR 13, 114] on the special region with three asymptotes which arose in connection with a problem on the inhomogeneous minimum of a binary cubic form with the three real linear factors.

J. H. H. Chalk (London).

Barnes, E. S. A problem of Oppenheim on quadratic forms. Proc. London Math. Soc. (3) 5, 167–184 (1955).

Let $f(x, y)=ax^2+bxy+cy^2$ be an indefinite real quadratic form with real coefficients and discriminant D. If $M(f)=\max\{a^2+\frac{1}{4}b^2, c^2+\frac{1}{4}b^2\}$, M is defined to be the lower bound of $M(f)$ where f runs through a complete class of equivalent forms. The author determines a constant q with the property that $M<qD$ unless the class of the forms which defines M contains a multiple of one of the forms x^2-3y^2, $3x^2+27xy-19y^2$, $6x^2+60xy-(\sqrt{(957)}+11)y^2$. For $\epsilon>0$ infinitely many nonequivalent forms exist for which $M>(q-\epsilon)D$. The proof depends on the Frobenius theory of infinite chains of reduced equivalent forms and their associated continued-fraction developments. The assumption $M\geqq qD$ is shown to imply that the continued-fraction development must be one of three types which correspond to the three classes of exceptional forms. *D. Derry.*

Sawyer, D. B. The lattice determinants of asymmetrical convex regions. II. Proc. London Math. Soc. (3) 5, 197–218 (1955).

[For part I see J. London Math. Soc. 29, 251–254 (1954); MR 15, 780.] Let K be a plane convex region of which the origin O is an interior point. Let $\Delta(K)$, $A(K)$ denote the lattice determinant and area of K respectively. The coefficient of symmetry λ of K is defined to be the upper bound of PO/OP' where P, P' are the boundary points of K on a chord POP'. Where $\lambda_0=(8+\sqrt{15})/7$,

$$\begin{aligned}
\gamma(\lambda)&=3\lambda^2-2\lambda+3-2(\lambda-1)(2\lambda^2+2)^{1/2} && \text{for } 1\leqq\lambda\leqq\lambda_0,\\
\gamma(\lambda)&=\tfrac{8}{3}-\tfrac{2}{3}(\lambda-2)^2 && \text{for } \lambda_0\leqq\lambda\leqq 2,\\
\gamma(\lambda)&=\tfrac{1}{2}(\lambda+1)^2/\lambda-1 && \text{for } 2\leqq\lambda\leqq 3,\\
\gamma(\lambda)&=2\lambda-(2\lambda^2-4\lambda-2)^{1/2} && \text{for } 3\leqq\lambda.
\end{aligned}$$

The purpose of the paper is to prove $A(K)\leqq\gamma(\lambda)\Delta(K)$. This is a generalization of the Minkowski result for symmetric regions for which $\lambda=1$. The problem is divided into a considerable number of subcases. The work is elementary but too detailed to be described in a review. The regions K for which $A(K)=\gamma(\lambda)\Delta(K)$ are shown to be convex polygons and are completely determined. *D. Derry.*

Sawyer, D. B. The lattice determinants of asymmetrical convex regions. III. Quart. J. Math., Oxford Ser. (2) 6, 27–33 (1955).

Let K be a convex body in Euclidean n-space symmetric with respect to one of its points, and of which the origin O is an interior point. The coefficient of symmetry λ of K is defined to be the upper bound of PO/OP', where P, P' are the boundary points of K on a chord POP'. Where λ_0 is defined as a root of a certain equation $q(\lambda)=2\lambda(\lambda+1)^{-1}$ for $1\leqq\lambda_0\leqq\lambda_0$, and $q(\lambda)=2^n\{1+n(\lambda-1)\}(\lambda+1)^{-n}$ for $\lambda_0\leqq\lambda$. The principal result of the above paper is that a convex body L exists within K, symmetric with respect to the origin, for which $V(L)\geqq q(\lambda)V(K)$, where $V(L)$, $V(K)$ denote the volumes of L and K respectively. This is a best possible result, as for each λ regions K and L are shown to exist for which $V(L)=q(\lambda)V(K)$. *D. Derry* (Vancouver, B. C.).

Obrechkoff, N. Sur l'approximation des formes linéaires. Bŭlgar. Akad. Nauk. Izv. Mat. Inst. 1, no. 2, 35–46 (1954). (Bulgarian. Russian and French summaries)

Let $\omega_1, \cdots, \omega_k$ be any k real numbers $(k\geqq 2)$, and let n be a positive integer. The author proves that there exist integers $x_1, \cdots, x_k$, not all zero, satisfying

$$|x_i|\leqq n\quad(i=1, \cdots, k),\quad |\omega_1x_1+\cdots+\omega_kx_k|\leqq\frac{n\Omega}{(n+1)^k-1},$$

where $\Omega=|\omega_1|+\cdots+|\omega_k|$. He also shows that the second inequality can be strict unless $\omega_1, \cdots, \omega_k$ have values proportional to ±1, $\pm(n+1)$, $\cdots$, $\pm(n+1)^{k-1}$ in some order. Both results are easily proved. *H. Davenport.*

Müller, Max. Über die Approximation reeller Zahlen durch die Näherungsbrüche ihres regelmässigen Kettenbruches. Arch. Math. 6, 253–258 (1955).

Let $z=[a_0, a_1, \cdots]$ denote the continued fraction development of the real number z, and let A_ν/B_ν denote its νth convergent. The author establishes several inequalities of which the following is typical. If $n\geqq 1$ and the continued fraction has at least $n+2$ elements, then

$$\left|z-\frac{A_\nu}{B_\nu}\right|<\frac{1}{(a^2_{n+1}+1)^{1/2}B_\nu^2}$$

for at least one of the values $\nu=n-1$, n, $n+1$. Various classical inequalities are deducible from his results [see, e.g., J. F. Koksma, Diophantische Approximationen, Springer, Berlin, 1936, Kap. III, §3, Satz 18, 21].

J. H. H. Chalk (London).

Perron, Oskar. Neue Periodizitätsbeweise für die regelmässigen und halbregelmässigen Kettenbrüche quadratischer Irrationalzahlen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1954, 321–333 (1955).

The author discusses various proofs of Lagrange's theorem on periodicity of the regular continued fraction for a quadratic irrational number, especially those of Gonçalves [Curso de álgebra superior, v. 1, Lisboa, 1953, p. 125; Univ. Lisboa. Revista Fac. Ci. A. (2) 2, 297–335 (1952), esp. p. 304; MR 16, 18], and gives new proofs of two inequalities of Gonçalves'. Analogous inequalities are obtained for semiregular continued fractions and hence simple proofs of periodicity. In this connection, a remark on p. 159 and discussion in §46 of the author's book, Die Lehre von den Kettenbrüchen [3rd ed., vol. 1, Teubner, Stuttgart, 1954; MR 16, 239], are supplemented. *H. S. Wall.*

ANALYSIS

Surányi, János. On the solvability of systems of linear inequalities. Acta Sci. Math. Szeged 16, 92–102 (1955).

The author establishes criteria for consistency of finite systems of linear inequalities (homogeneous and non-homogeneous) expressed in terms of the coefficient matrix, previously obtained by Motzkin [Thesis, Basel, 1936] and the reviewer [Pacific J. Math. 3, 523–530 (1952); MR 14, 541].

L. M. Blumenthal (Columbia, Mo.).

Chassan, J. B. A statistical derivation of a pair of trigonometric inequalities. Amer. Math. Monthly 62, 353–356 (1955).

The author establishes the inequality

$$\frac{\sum_i\cos^2\theta_i}{\sum\sum_{j>i}\sin^2(\theta_i-\theta_j)}\leqq\binom{n}{2}^{-2}\sum_{i=1}^{n}\left(\sum_{j\neq i}\frac{\cos\theta_j}{\sin(\theta_i-\theta_j)}\right)^2,$$

where $0\leqq\theta_i<\pi$ and $\theta_i\neq\theta_j$ for $i\neq j$ $(i, j=1, \cdots, n)$, and also

the similar inequality with sines instead of cosines in the numerators, by comparing the variances of a pair of minimum-variance estimators with those of certain less efficient estimators. The first pair estimate the coordinates of a fixed point in a plane by the method of least squares applied to line-of-sight observations, while the second pair is obtained by averaging the position-vectors of all intersections of lines of sight. *H. P. Mulholland* (Birmingham).

Tandori, Károly. Über die Konvergenz singulärer Integrale. Acta Sci. Math. Szeged **15**, 223–230 (1954).

Theorem: Let $p\geqq 1$, $q=p/(p-1)$, $\varphi_n(t)$ $(n=1, 2, \cdots)$ be measurable in (0, 1) and denote by $L_0{}^p$ the class of functions $f(t)\in L^p(0, 1)$ for which $\int_0{}^h|f(t)|^p dt=o(h)$ as $h\downarrow 0$. Then (*) $\lim_{n\to\infty}\int_0{}^1 f(t)\varphi_n(t)dt=0$ holds for every $f(t)\in L_0{}^p$ if and only if the sequence

$$\sum_{m=0}^{\infty}2^{-m/p}\left(\int_{2^{-m-1}}^{2^{-m}}|\varphi_n(t)|^q dt\right)^{1/q}\quad(n=1, 2, \cdots)$$

is bounded and $\lim_{n\to\infty}\int_0{}^\eta\varphi_n(t)dt=1$ for every $0<\eta\leqq 1$. Other necessary and sufficient conditions for (*) were given by D. Faddeev [Mat. Sb. N.S. **1**(43), 351–368 (1936)] in the case $p=1$ and by B. I. Korenblyum, S. G. Kreĭn and B. Ya. Levin [Dokl. Akad. Nauk SSSR (N.S.) **62**, 17–20 (1948); MR **10**, 306] in the general case. The author shows how Faddeev's conditions can be obtained from his own. *A. Dvoretzky* (New York, N. Y.).

Krylov, V. I. Increasing the accuracy of mechanical quadratures when the main part of the integration is over a small interval in the integral representation of the remainder of the quadrature. Dokl. Akad. Nauk SSSR (N.S.) **101**, 989–991 (1955). (Russian)

The author is concerned with quadrature formulas of the form

$$\int_a^b p(x)f(x)dx=\sum_{k=1}^n A_k f(x_k)+\int_a^b f^{(m)}(x)K(x)dx,$$

in the case in which there is a subinterval outside which $K(x)$ is small. In order to exhibit the greater influence of the rest of the interval, he takes α_0 as the abscissa of the centroid of K, puts

$$\int_a^b f^{(m)}(x)K(x)dx=f^{(m)}(\alpha_0)\int_a^b K(x)dx+\int_a^b f^{(m+2)}(x)K_1(x)dx,$$

with a modified kernel K_1, and iterates this process. He illustrates the modified formula with some explicit calculations for the case when $p(x)=(1-x)^p(1+x)^q$. *R. P. Boas, Jr.* (Evanston, Ill.).

Pastides, Nicolas. On the classes $C\{M_n\}$ of infinitely differentiable real functions. J. London Math. Soc. **30**, 212–220 (1955).

L'auteur donne une condition nécessaire et suffisante, portant sur les quantités $k_{p,n}=\max_{q\leq n}\{\max_{x\in I}|f_p{}^{(q)}/M_q|^{1/q}\}$, pour qu'une suite de fonctions $\{f_p(x)\}$, satisfaisant aux inégalités $|f_p{}^{(n)}(x)|\leqq k_p{}^n M_n$ sur $I\equiv[a, b]$, admette, sur I, une limite (lorsque celle-ci existe) appartenant à la classe $C\{M_n\}$. *S. Mandelbrojt* (Houston, Texas).

Džrbašyan, M. M. On two quasi-analytic classes of functions on the real axis. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **8**, no. 1, 3–14 (1955). (Russian. Armenian summary)

The author constructs two quasi-analytic classes whose definitions involve best approximation. (1) Let $A_\sigma f$ denote the best approximation to $f(x)$ by entire functions of exponential type σ. As S. Bernstein has shown [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **51**, 487–490 (1946) ⊂ Sobranie Sočineniĭ, vol. 2, no. 84 (1954); MR **8**, 20; **16**, 433], the inequality (*) $A_\sigma f\leqq Be^{-b\sigma}$ for all positive σ implies that $f(x)$ coincides on the real axis with a function which is regular and bounded in $|y|<b$. The author shows that if f is bounded on the real axis and if (*) holds for a sequence $\sigma_n\uparrow\infty$ then $f(x)\equiv 0$ on some real interval implies $f(x)\equiv 0$ for all real x. (2) Let λ_k and μ_k be complex numbers with bounded real parts, and $\alpha_k=\Im(\lambda_k)\geqq c>0$, $\beta_k=-\Im(\mu_k)\geqq c>0$, $\sum 1/\alpha_k=\sum 1/\beta_k=\infty$, $U_{n,m}=\min\{\sum_1^n 1/\alpha_k, \sum_1^m 1/\beta_k\}$. Let $f(x)$ be bounded on the real axis and let there exist, for $n\geqq 1$ and $m\geqq 1$, rational functions

$$R_{n,m}(z)=P_{n+m}(z)\Big/\left\{\prod_{k=1}^n(z-\alpha_k)\prod_{k=1}^m(z-\beta_k)\right\}$$

such that (**) $\sup_x|f(x)-R_{n,m}(x)|\leqq A\exp\{-bU_{n,m}\}$. Then $f(x)$ coincides with a function which is regular and bounded in a strip $|y|<a$. If (**) holds only for sequences m_k and n_k, and if further the expression $\{\sum_{k=1}^{n_k}1/\alpha_k\}/\{\sum_{k=1}^{m_k}1/\beta_k\}$ is bounded and bounded away from 0, then $f(x)\equiv 0$ on some real interval implies $f(x)\equiv 0$ for all real x.
R. P. Boas, Jr. (Evanston, Ill.).

Theory of Sets, Theory of Functions of Real Variables

Cuesta, N. Ordinal algebra. Rev. Acad. Ci. Madrid **48**, 103–145 (1954). (Spanish)
Cuesta, N. Ordinal arrangement. Rev. Mat. Hisp.-Amer. (4) **14**, 237–268 (1954). (Spanish)

The author classifies general binary relations, concentrating on partial and total orders. He obtains extensions of such orders by inserting new elements into various "cavities" which, in the case of a total order, are given by the Dedekind decompositions (as well as those with an empty lower or upper class) of the ordered set. This enables him to construct increasing sequences of order types. He concludes with an examination of some of Denjoy's [L'énumération transfinie, livre I, Gauthier-Villars, Paris, 1946; MR **8**, 254] criticisms of Cantor. *F. Bagemihl.*

de Barros Neto, José. On the construction of a completely additive class. Soc. Parana. Mat. Annuário **1**, 9–11 (1954). (Portuguese)

Il s'agit d'un article d'exposition, sur la notion de classe complètement additive d'ensembles (σ-corps) et sur la construction de la classe des boréliens, par induction transfinie, en partant de la classe des ouverts.
J. Sebastião e Silva (Lisbonne).

Denjoy, Arnaud. Les points inflexionnels. C. R. Acad. Sci. Paris **238**, 2469–2472 (1954).

A point M of a continuum or a perfect pointset Γ in the plane is called a point of inflection if there exists a tangent to Γ at M and a number $\rho>0$ satisfying the following condition: If the segment of the tangent of length 2ρ and centered at M is rotated about M by $\pi/6$, then the double sector swept out by the diameter contains all points of Γ that are not outside the circle of radius ρ about M. The set of all points of inflection of Γ is an $F_{\sigma\delta}$. In many cases, the set of all inflectional directions has the measure 0. The author conjectures that this is a general law. The concept of point of inflection can be extended to sets in Cartesian spaces of higher dimension. *K. Menger* (Chicago, Ill.).

Nagami, Keiô. Baire sets, Borel sets and some typical semi-continuous functions. Nagoya Math. J. 7, 85–93 (1954).

The following properties of a real-valued function g on a topological space are shown to be equivalent: (1) $g(x)=\lim g_n(x)$, g_n continuous, $g_n(x)\geqq g_{n+1}(x)$; (2) every set $\{x|g(x)\geqq\lambda\}$ is "elementary-closed", i.e. of the form $\{x|\varphi(x)\geqq 0\}$, φ continuous. It is proved that every countably compact Baire set (in an arbitrary topological space) is "elementary-closed". A proof of the following insertion theorem is given: if g and $-h$ possess property (1), $g(x)<h(x)$, then there exists a continuous f with

$$g(x)<f(x)<h(x).$$

[Reviewer's note: It is easy to prove, using the lemma given on p. 89, a stronger result: if g, h are Baire functions, g upper semicontinuous, h lower semicontinuous, $g(x)<h(x)$, then there exists a continuous f with $g(x)<f(x)<h(x)$.]

M. Katětov (Prague).

Meyer, Burnett. On restricted functions. Amer. Math. Monthly 62, 29–30 (1955).

A concept of restrictedness is defined which is similar to cliquishness of H. Thielman [Proc. Iowa Acad. Sci. 59, 338–343 (1952); MR 14, 628]. The function $f(x)$, defined on the set E, is said to be restricted at the point a, if a is a limit point of E and if $\lim\sup_{x\to a} f(x)$ and $\lim\inf_{x\to a} f(x)$ are both finite. It is shown that if f is restricted on $E'\subset E$ and unrestricted on $E''=E-E'$, and if E' is everywhere dense in E, then E'' is nowhere dense. *M. Cotlar.*

Džvaršeĭšvili, A. G. On generalized absolutely continuous functions of two variables. Soobšč. Akad. Nauk Gruzin. SSR 15, 129–133 (1954). (Russian)

Let the function $F(x,y)$ be defined on the rectangle $R_0=[(a,b)(c,d)]$, and let F be ACG in the sense of Čelidze [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 15, 155–242 (1947); MR 14, 735] on R_0. Then: (i) $R_0=\sum_1^\infty A_k$, where the A_k are closed sets, and on each A_k there exists the λ-regular derivative $D^{(\lambda)}_{A_k}F$, which is L-integrable on A_k. (ii) $R_0=\sum_1^\infty A_k+H$ ($|H|=0$) and on each A_k there exists the strong derivative $D^s{}_{A_k}F$, which is integrable on A_k.

(iii) $\partial^2{}_{ap}F(x,y)/\partial x\partial y=\partial^2{}_{ap}F(x,y)/\partial y\partial x=D_{ap}F(x,y)$.

Besides, if $F(x,y)$ is an arbitrary continuous function on R_0, then $R_0=\sum_1^\infty A_k+H$ ($|H|=0$) and $D^s{}_{A_k}F$ exists on each A_k, if, and only if, $R_0=\sum_1^\infty E_k+H'$ ($|H'|=0$) and $F\in$ (Lip) on each E_k. Finally some properties concerning double integrals of Denjoy-Čelidze are indicated, similar to previous results of the author [Soobšč. Akad. Nauk Gruzin. SSR 14, 393–398 (1953); MR 16, 345]. No proofs are given.

M. Cotlar (Mendoza).

Roşculeţ, Marcel N. Sur les dérivées partielles polydimensionnelles orientées. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6, 811–818 (1954). (Romanian. Russian and French summaries)

Considérant les dérivées polydimensionnelles orientées, l'auteur obtient des résultats plus generaux en utilisant un polytope à variétés quelconques, au lieu du polytope à variétés linéaires. On obtient ainsi une classe plus étendue de variétés, pour lesquelles la dérivée polydimensionnelle est obtenue à l'aide des dérivées successives suivant le-directions des tangentes aux certaines courbes paramés triques. (Author's summary.) *M. Reade.*

Austin, Donald G. An isomorphism theorem for finitely additive measures. Proc. Amer. Math. Soc. 6, 205–208 (1955).

This paper is a contribution to the classification of finitely additive measures ν on a countable set X which are non-atomic and normalized by $\nu(X)=1$. A class of such measures [Jordan measures] may be obtained by defining a 1-to-1 map T of X onto a dense subset D of $[0,1]$, and choosing $\nu(A)$ to be the Jordan content of the closure of $T(A)$. In 1946, the reviewer introduced a special measure μ on the set I of positive integers [Amer. J. Math. 68, 560–580 (1946); MR 8, 255]. In a subsequent paper [ibid. 69, 413–420 (1947); MR 8, 506], Ellen Buck and the reviewer showed in theorem 1 that μ is a universal model in the sense that every separable measure ν is isomorphic to a contraction of μ, under a suitable 1-to-1 map of X onto I; furthermore, theorem 2 of that paper implied that μ in turn is isomorphic to a particular Jordan measure. Thus, the Jordan measures also form a universal model for the measures ν which are separable. In the paper under review, the author establishes this directly, producing the desired mapping T of X onto a dense set D in $[0,1]$. He also points out some of the implications of this, brought about by the special properties of Jordan content. *R. C. Buck.*

Marczewski, E. Remarks on the convergence of measurable sets and measurable functions. Colloq. Math. 3, 118–124 (1955).

A typical theorem is that for a finite measure space convergence in measure is equivalent to convergence almost everywhere if and only if the measure is purely atomic. In connection with the pertinent study of atomic measures, the author proves that a necessary and sufficient condition for the existence of a purely atomic probability measure m and the existence of a sequence $\{E_n\}$ of stochastically independent sets such that $m(E_n)=p_n$ is that $\sum_{n=1}^\infty \min(p_n, 1-p_n)<\infty$. *P. R. Halmos.*

Phakadze, Š. S. On sets absolutely of measure zero. Soobšč. Akad. Nauk Gruzin. SSR 15, 201–205 (1954). (Russian)

The author considers congruence-invariant σ-algebras containing the unit cube in a finite-dimensional Euclidean space. Such an algebra is called solvable if it admits a congruence-invariant, non-negative, and countably additive measure that takes the value 1 on the unit cube. A set X is absolutely of measure zero if whenever Y is a countable union of sets, each of which is congruent to a subset of X, then there exists a solvable algebra containing Y, and if, moreover, $m(Y)=0$ for every measure m (satisfying the above conditions) on such an algebra. The main purpose of the paper is to state (without proofs) some necessary and sufficient conditions that a set be absolutely of measure zero. A typical theorem is phrased in terms of the author's concept of comprehensive sets. A set A is comprehensive if there exists a set B of (Lebesgue) measure zero such that the complement of B is a countable union of sets each of which is congruent to a subset of A. A necessary and sufficient condition that X be absolutely of measure zero is that $A\cup X$ be non-comprehensive whenever A is non-comprehensive. *P. R. Halmos* (Chicago, Ill.).

Tsuchikura, Tamotsu. Remarques sur les sommes riemanniennes. J. Math., Tokyo 1, 155–160 (1953).

Utilisant des résultats de Offord [Fund. Math. 35, 259–270 (1948); MR 10, 248], l'auteur indique des conditions

pour que $F_n(x)=n^{-1}\sum_{k=1}^{n-1} f(x+kn^{-1})$ tende vers $\int_0^1 f(t)\,dt$ quand $n\to\infty$, ou pour que $F_n(x)$ soit sommable (C, α) vers la même somme. Puis il donne un exemple de fonction f non mesurable telle que $F_n(x)\to 0$ quand $n\to\infty$, quel que soit x. *J. P. Kahane* (Montpellier).

Dias Agudo, Fernando Roldão. Primitivability of functions of a real variable. Ciência, Lisboa **2**, nos. 11–12, 17–23 (1955). (Portuguese)

Expository paper.

Theory of Functions of Complex Variables

Talmanov, A. D. Remark on the paper of V. S. Fëdorov, "The works of N. N. Luzin on the theory of functions of a complex variable." Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 167–168 (1955). (Russian)

Among the unsolved Problems left in the notes of the late N. N. Lusin [cf. V. S. Fëdorov, Uspehi Mat. Nauk (N.S.) **7**, no. 2(48), 7–16 (1952); MR **13**, 810] was the following: Is it possible to continue a complex-valued function $f(x)=u(x)+iv(x)$, which is differentiable on a segment $[a, b]$ of the real axis, into a region of the z-plane containing $[a, b]$ in such a way that the extended function $F(z)$ will be differentiable, as a function of the complex variable $z=x+iy$, at every point of $[a, b]$? The author shows that such an extension is possible in wide variety of ways; a particularly simple extension is given by means of the function $F(z)=(1+i)^{-1}[f(x-y)+if(x+y)]$. *A. J. Lohwater* (Helsinki).

Erdös, P., Herzog, F., and Piranian, G. Polynomials whose zeros lie on the unit circle. Duke Math. J. **22**, 347–351 (1955).

The authors study polynomials of the type

$$P(z)=\prod_1^n (1-z/\omega_j),$$

where $|\omega_j|=1$ for all j. By considering $\log P(z)$ and suitably choosing the integers k_j in the form $\prod_1^\infty [1+(z/\omega_j)^j]^{k_j}$, they prove the existence of a polynomial such there are two points z' and z'' on every radius of the unit disc with $|P(z')|<1$, $|P(z'')|>1$. They show further that, for every polynomial of type $P(z)$ but of degree $n\leqq 4$, there are two directions θ' and θ'' such that $|P(re^{i\theta'})|\leqq|1-r^n|$, $|P(re^{i\theta''})|\geqq 1+r^n$. *M. Marden* (Milwaukee, Wis.).

Mac Lane, Gerald R. On a conjecture of Erdös, Herzog, and Piranian. Michigan Math. J. **2** (1953–54), 147–148 (1955).

In the paper reviewed above the authors ask whether there exists a universal constant L such that for every polynomial $P(z)$ whose zeros all lie on the unit circle, there is a path of length not exceeding L from the origin to the unit circle along which path $|P(z)|\leqq 1$. In the present paper, MacLane proves the answer to this question is in the negative. He shows that for each n there is a best possible $L=L_n$ but that $L_n\to\infty$. *M. Marden* (Milwaukee, Wis.).

4 ***Nehari, Zeev. Univalent functions and linear differential equations.** Lectures on functions of a complex variable, pp. 49–60. The University of Michigan Press, Ann Arbor, 1955. $10.00.

The basic idea is the observation that if $f(z)=u(z)/v(z)$ is the quotient of two analytic functions univalent in a domain D and if $p(z)=\frac{1}{2}\{f(z), z\}$, where $\{w, z\}$ is the Schwarzian derivative, then each solution $C_1u(z)+C_2v(z)$ of the differential equation $y''+p(z)y=0$ has at most one zero in D and vice versa. This observation had earlier led the author to criteria of univalence in the unit circle [Bull. Amer. Math. Soc. **55**, 545–551 (1949); MR **10**, 696]. By related methods he now proves that a single-valued $f(z)$ is univalent in $|z|<R$ if there exists a γ, $-\frac{1}{2}\pi<\gamma<\frac{1}{2}\pi$, such that $\Re(e^{i\gamma}[\frac{1}{2}+z^2\{f(z), z\}])>0$, $|z|<R$. Here $\frac{1}{2}$ is the best possible constant since the bracket is zero for $f(z)=z^2$. This result is also valid for an annulus. Using the associated integral equation and the Green's function, he proves that if $f(z)$ is holomorphic in $|z|<1$ and $\int_0^{2\pi}|\{f(z), z\}|\,d\theta\leqq 8$, $z=e^{i\theta}$, then $f(z)$ is univalent in $|z|<1$. Similarly, if $\int_0^{2\pi}|z^2\{f(z), z\}-\frac{1}{2}|\,d\theta<16/\pi$, $z=re^{i\theta}$, $0<r<R$, then $f(z)$ is univalent in $|z|<R$. Finally, if $f(z)$ is regular and single-valued on $|z|=r$ and there takes on a particular value n times, then

$$n^2\leqq\tfrac{1}{4}\pi\int_0^{2\pi}|z^2\{f(z), z\}-\tfrac{1}{2}|\,d\theta,\quad z=re^{i\theta}.$$

If in addition $f'(z)\neq 0$, then

$$n^2\leqq 1-2\min_{|z|=r}\Re[z^2\{f(z), z\}].$$

E. Hille (New Haven, Conn.).

Lohin, I. F. On the representation of analytic functions by Faber polynomials. Mat. Sb. N.S. **36(78)**, 441–444 (1955). (Russian)

Definition of an analogue of the Mittag-Leffler star for analytic functions $f(z)$ defined in a bounded, simply connected domain D with connected complement and construction of a triangular matrix method of summation summing the expansion of $f(z)$ in Faber polynomials of D for all z in the Mittag-Leffler star. *W. H. J. Fuchs.*

Kuz'mina, A. L. On series of orthogonal polynomials. Ukrain. Mat. Ž. **6**, 363–366 (1954). (Russian)

Let C be a simple closed, analytic curve in the ζ-plane, $\zeta=\phi(z)$ the function mapping $|z|>1$ conformally on the outside of C, normalized by $\phi(\infty)=\infty$, $\phi'(\infty)>0$, $\phi(z)$ is regular in a domain $|z|>\rho$, $\rho<1$. Let $D(r)$ be the image of $|z|>r$ by $\phi(z)$. Let $w(\zeta)=|q(\zeta)|^2$ be a weight-function on C, where q is regular and $\neq 0$ in $D(\rho)$. Let $\{p_n(\zeta)\}$ be the polynomials obtained by orthogonalizing $\{1, \zeta, \zeta^2, \cdots\}$ on C with respect to the weight $w(\zeta)$. The author considers series $\sum c_np_n(\zeta)$ and proves analogues of theorems on the convergence of power series. Basic is the transfer-principle: Regarding convergence the series $\sum c_np_n(\zeta)$ and $\sum c_nz^n$ show the same behavior at points inside and on the radius of convergence of the power series $(\zeta=\phi(z))$. *W. H. J. Fuchs* (Ithaca, N. Y.).

Leont'ev, A. F. On the region of boundedness of a sequence of Dirichlet polynomials. Mat. Sb. N.S. **35(77)**, 175–186 (1954). (Russian)

L'auteur démontre le théorème suivant: soit $\varphi(z)=\sum c_nz^n$ une fonction entière dont la croissance ne dépasse pas celle d'une fonction entière d'ordre un du type minimum, et désignons par $M(F)$ l'opérateur $\sum c_nF^{(n)}(z)$; soit $\{f_n(z)\}$ une suite de fonctions entières telles que $M(f_n)=0$ $(n\geqq 1)$. Désignons par G l'ensemble de tous les points P jouissant de la propriété suivante: dans un voisinage suffisamment petit de P, la suite $|f_n(z)|$ est bornée. La conclusion est alors que toute partie connexe de G est convexe. En particulier,

soit $\{\lambda_j\}$ une suite complexe tendant vers l'infini, avec $|\lambda_j| \leqq |\lambda_{j+1}|$. En désignant G par l'ensemble des points P tels que au voisinage de chaque P la suite des polygones (1) $P_n(s) = \sum_{j=1}^n a_{n,j} e^{\lambda_j s}$ est bornée, toute partie connexe de G est convexe, à condition que $\lim n/\lambda_n = 0$. Cette dernière condition ne peut pas être améliorée, ce qui doit être compris dans le sens suivant: à tout $\epsilon > 0$ on peut faire correspondre une suite $P_n(z)$ de la forme (1) avec $\lim n/|\lambda_n| = \epsilon$, pour laquelle la conclusion concernant G est en defaut. Dans le cas où $f_n = P_n$ (donné par (1), l'opérateur M est formé à partir de $\varphi(z) = \prod(1 - z^2\lambda_n^{-2})$. L'auteur tire de son théorème principal quelques conclusions concernant la forme du domaine D tel qu'une suite $f_n(z)$, satisfaisant à la condition $M(f_n) = 0$, converge uniformément dans le voisinage de chaque point de D. Peu de points nouveaux dans les démonstrations; celles-ci diffèrent peu de celles utilisées par Leont'ev [Trudy Mat. Inst. Steklov. 39 (1951); MR 14, 1074]. *S. Mandelbrojt* (Houston, Tex.).

Ostrowski, Alexander. Über die analytische Fortsetzung von Taylorschen und Dirichletschen Reihen. Math. Ann. 129, 1–43 (1955).

L'auteur généralise le théorème bien connu de Cramér-Ostrowski concernant le prolongement analytique d'une série de la forme (1) $F(s) = \sum \varphi(\lambda_\nu) c_\nu e^{-\lambda_\nu s}$. Des généralisations sont connues [Soula, J. Math. Pures Appl. (9) 4, 339–353 (1925); V. Bernstein, Leçons sur les progrès recents de la théorie des séries de Dirichlet, Gauthier-Villars, Paris, 1933], où, au lieu de supposer que $\varphi(z)$ est une fonction entière, satisfaisant, pour chaque $\epsilon > 0$, à la condition que (2) $\varphi(z) = O(e^{(k+\epsilon)r})$ $(k \geqq 0)$, on suppose seulement que cette fonction est holomorphe pour (3) $|z| \geqq R > 0$, $\alpha_1 < \arg z < \alpha_2$, $\varphi(z)$ satisfaisant aussi à la condition (2) dans tout angle (4) $\alpha_1 + \epsilon \leqq \arg z \leqq \alpha_2 - \epsilon$. Dans Soula et V. Bernstein on suppose que $\frac{1}{2}\pi > \alpha_2 > 0 > \alpha_1 > -\frac{1}{2}\pi$. L'auteur démontre maintenant le théorème suivant: Soit (5) $f(s) = \int_a^\infty e^{-\lambda s} dA(\lambda)$, $A(\lambda)$ étant localement à variation bornée, l'abscisse de convergence de (5), σ^*, étant finie. Soit: $\alpha_1 < 0 < \alpha_2$, $\alpha_2 - \alpha_1 \leqq \pi$, et soit $\varphi(z)$ une fonction holomorphe dans (3), satisfaisant à (2) dans chaque angle (4). Posons

$$(6) \qquad F(s) = \int^\infty e^{-\lambda s} \varphi(\lambda) dA(\lambda).$$

$F(s)$ est alors holomorphe pour $\sigma > \sigma^* + k$ $(s = \sigma + it)$; et, si $f(s)$ peut être prolongée jusqu'au point s_0 suivant toutes les directions α de l'intervalle $[\pi/2 - \alpha_1, 3\pi/2 - \alpha_2]$ à partir du demi-plan $\sigma > \sigma^* + k$, avec le rayon d'holomorphie autour de chaque point du segment, suivant lequel on prolonge, supérieur à k, $F(s)$ est prolongeable en s_0 suivant les mêmes directions à partir du demi-plan $\sigma > \sigma^* + k$. Un théorème est aussi donné pour le cas $\alpha_2 - \alpha_1 > \pi$. *S. Mandelbrojt.*

Akaza, Tôru. On the remark of Laasonen's theorem. Sci. Rep. Kanazawa Univ. 2, no. 2, 1–6 (1954).

This paper investigates conditions under which the series expansion for a certain Fuchsoid function given by Laasonen [Ann. Acad. Sci. Fenn. Ser. A. I. no. 25 (1944); MR 8, 24] is absolutely convergent. *M. H. Heins.*

Džrbašyan, M. M., and Tavadyan, A. B. Some extremal problems for entire functions. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki 7, no. 5, 1–17 (1954). (Russian. Armenian summary)

The authors consider the class W_σ of entire functions f of exponential type σ, belonging to L^2 on the real axis, and the subclasses $W_\sigma\{a_{2p-2}; 0\}$, $W_\sigma\{0; a_{2q-1}\}$, $W_\sigma\{a_{2p-2}; a_{2q-1}\}$ for which respectively $f^{(2k)}(0) = a_{2k}$ $(k = 0, 1, \cdots, p-1)$; $f^{(2k+1)}(0) = a_{2k+1}$ $(k = 0, \cdots, q-1)$; or both of these hold. They determine the minimum of the L^2 norm of f for each class. The extremal functions are expressible explicitly in terms of Bessel functions and the a's. As a consequence the authors find a necessary and sufficient condition for the existence of an f in W_σ for which $f^{(k)}(0) = a_k$ $(k = 0, 1, \cdots)$. They also consider similar problems for functions of order $\frac{1}{2}$ and finite type with $f(x)x^{-1/4}$ belonging to $L^2(0, \infty)$. *R. P. Boas, Jr.* (Evanston, Ill.).

***Hayman, W. K. The growth of entire and subharmonic functions.** Lectures on functions of a complex variable, pp. 187–198. The University of Michigan Press, Ann Arbor, 1955. \$10.00.

The author gives a survey of the developments of the past decade concerning the growth of entire functions and functions subharmonic in the finite plane. Among the topics discussed are the author's answer to the Wiman conjecture and his example of an entire function with a defective value that is neither asymptotic nor invariant under change of origin. *M. H. Heins* (Providence, R. I.).

Sales Vallés, Francisco de A. On entire functions of irregular growth. Collect. Math. 7, 69–94 (1954). (Spanish)

There is a conventional way of attaching an "order" to any logarithmico-exponential function. The author considers functions f such that $\limsup f(x)/\Phi(x) = L$, $\liminf f(x)/\Psi(x) = l$, L and l different from 0 and ∞, with Φ and Ψ logarithmico-exponential functions; he then calls the order of Φ the upper order of f, the order of Ψ the lower order of f, and the order of Φ/Ψ the oscillation of the increase of f, and develops a calculus of these quantities. He establishes inequalities connecting these orders with the coefficients in the power series of f, expresses the rate of growth of a function in various directions by means of them, and makes other remarks. *R. P. Boas, Jr.*

Clunie, J. Note on integral functions of infinite order. Quart. J. Math. Oxford Ser. (2) 6, 88–90 (1955).

The author generalizes results of Shah [same J. (2) 1, 112–116 (1950); MR 12, 16] and Shah and Khanna [Math. Student 21, 47–48 (1953); MR 15, 207] by showing that for an entire function f of infinite order

$$\liminf \frac{\log \{r^p M(r, f^{(p)})\}}{N(r, f)} = 0,$$

where M denotes maximum modulus, N denotes central index, and p is any function of N such that $p(N) = o(N/\log N)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Fišman, K. M. On the integral representation of certain classes of entire functions. Uspehi Mat. Nauk 10, no. 2(64), 187–194 (1955). (Russian)

The author considers a continuous positive decreasing function $\sigma(\rho)$ such that $\alpha_n = \int_0^\infty \sigma(\rho)\rho^n d\rho < \infty$ and $\alpha_{2n}^{1/n} \to \infty$. He defines L_σ^2 as the Hilbert space of functions $\phi(z)$ such that $\int_0^\infty \int_0^{2\pi} |\phi(z)|^2 \sigma(\rho)\, d\rho d\theta < \infty$, and Z_σ^2 as the class of entire functions in L_σ^2. He shows that Z_σ^2 is a Hilbert space and that the sequence $z^n(2\pi\alpha_{2n})^{-1/2}$ is a basis for it. His main result is that Z_σ^2 also is the class of $f(z)$ with the representation

$$f(z) = \int_0^\infty \int_0^{2\pi} \phi(w) E(\bar{w}z) \sigma(\rho) d\rho d\theta \quad (w = \rho e^{i\theta}),$$

where

$$\int_0^\infty \int_0^{2\pi} |\phi(w)|^2 \sigma(\rho) d\rho d\theta < \infty$$

and $2\pi E(z) = \sum_{n=0}^\infty z^n/\alpha^{2n}$. There are further remarks about the class Z_σ^2. *R. P. Boas, Jr.* (Evanston, Ill.).

Gol'dberg, A. A. On a problem in the theory of distribution of values of meromorphic functions. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 3–5 (1954). (Ukrainian. Russian summary)

Announcement of the results of the paper reviewed below. *A. J. Lohwater* (Helsinki).

Gol'dberg, A. A. On the inverse problem of the theory of the distribution of the values of meromorphic functions. Ukrain. Mat. Ž. **6**, 385–397 (1954). (Russian)

The author constructs a class of Riemann surfaces having a finite number of "almost periodic" ends in order to solve the problem of finding a meromorphic function having given defects $\delta(a_k)$ on a given sequence of points $\{a_k\}$. The restrictions on the $\delta(a_k)$ are that (1) $\sum\delta(a_k) < 2$, and (2) all but a finite number of the $\delta(a_k)$ are zero. This result extends the work of Le Van Thiem [Comment. Math. Helv. **23**, 26–49 (1949); Ann. Sci. Ecole Norm. Sup. (3) **67**, 51–98 (1950); MR **11**, 22; **12**, 17], who required, in addition, that the $\delta(a_k)$ be rational. *A. J. Lohwater* (Helsinki).

Künzi, Hans. Zwei Beispiele zur Wertverteilungslehre. Math. Z. **62**, 94–98 (1955).

Examples of Streckenkomplexe which are of interest in the theory of the distribution of the values of a meromorphic function are constructed. The first example is of a Streckenkomplex having a doubly-periodic end endowed with a given asymmetric boundary. The second example shows how four "Viertelsenden" can be adjoined to a complex which has no logarithmic elementary domains. *M. H. Heins.*

Bagemihl, F. Curvilinear cluster sets of arbitrary functions. Proc. Nat. Acad. Sci. U. S. A. **41**, 379–382 (1955).

Let $K = (|z| = 1)$, $D = (|z| < 1)$. By an arc at $e^{i\theta}$ is meant a Jordan arc in D having one endpoint at $e^{i\theta}$ and the other in D. If Γ is an arc at $e^{i\theta}$ and f is an (arbitrary) map of D into the Riemann sphere, the cluster set of f at $e^{i\theta}$ along Γ, denoted by $C(f, \Gamma)$, is the set of $w \in R$ for which there exists a sequence $\{z_n\}$, $z_n \in \Gamma$, $z_n \to e^{i\theta}$ such that $f(z_n) \to w$. The author shows: I. If $S \subseteq D$, there exists $K^* \subseteq K$ with $K - K^*$ countable such that, for every $e^{i\theta} \in K^*$, if Γ_1 and Γ_2 are arcs at $e^{i\theta}$ either Γ_1 and Γ_2 both intersect S or both intersect $D - S$. II. There exists $K^* \subseteq K$ with $K - K^*$ countable such that for every $e^{i\theta} \in K^*$ and every pair of arcs Γ_1, Γ_2 at $e^{i\theta}$, $C(f, \Gamma_1) \cap C(f, \Gamma_2) \neq 0$. Applications of II to arbitrary f, continuous f, and meromorphic f are given. *M. H. Heins* (Providence, R. I.).

Belinskiĭ, P. P., and Gol'dberg, A. A. Application of a theorem on conformal mappings to questions of invariance of defects of meromorphic functions. Ukrain. Mat. Ž. **6**, 263–269 (1954). (Russian)

An example is given of a meromorphic function of finite order for which the defect is not invariant under a translation of the origin; this answers a question raised by Valiron [C. R. Acad. Sci. Paris **225**, 556–558 (1947); MR **9**, 139]. If E is some set in $1 < r < \infty$, $\chi(t, E)$ the characteristic function of E, and cE the complement of E with respect to $1 < r < \infty$, it is said that $E \in A$ if

$$\limsup\, (\log r)^{-1} \int_1^r \chi(t, E) dt < \infty.$$

It is proved that, for a meromorphic function of finite order, the quantities

$$\sup_{cE \in A} \liminf_{r \in E,\, r\to\infty} \frac{m(r,a)}{T(r)} \quad \text{and} \quad \inf_{cE \in A} \limsup_{r \in E,\, r\to\infty} \frac{m(r,a)}{T(r)}$$

are invariant under translation. It is also shown that, in order that the defect (in the sense of either Nevanlinna or Valiron) of a meromorphic function be invariant under a pseudo-analytic transformation, it is sufficient that either $0 < K_1 < T(r)/r^\alpha < K_2 < \infty$ or $0 < K_1 < N(r, a)/r^\alpha < K_2 < \infty$. *A. J. Lohwater* (Helsinki).

Heins, Maurice. Meromorphic functions with assigned asymptotic values. Duke Math. J. **22**, 353–356 (1955).

L'auteur construit une fonction analytique sur une surface de Riemann donnée F non compacte, et dont l'ensemble des valeurs asymptotiques est un ensemble analytique donné du plan complexe, contenant ou non le point à l'infini. Si F est le cercle-unité, l'existence d'une telle fonction a été démontrée par Kierst [Fund. Math. **27**, 226–233 (1936)]; si F est le plan entier une fonction entière satisfaisant à ces conditions a été construite par l'auteur [Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, Lund. Univ. Mat. Inst., 1954, pp. 56–60; MR **16**, 809]. Le problème actuel se ramène alors à la construction d'une fonction analytique sur F, et ne possédant aucune valeur asymptotique finie. En fait l'auteur construit une fonction analytique f sur F telle que pour toute région bornée Ω du plan fini toute composante de $f^{-1}(\Omega)$ soit relativement compacte. La démonstration utilise une exhaustion de F permettant l'application d'un théorème de Behnke et Stein [Math. Ann. **120**, 430–461 (1949), p. 456; MR **10**, 696] et d'un théorème de Garabedian précisé par l'auteur dans son article [Ann. of Math. (2) **52**, 568–573 (1950); MR **12**, 259] sur le problème d'interpolation de Pick-Nevanlima. *L. Fourès.*

Künzi, Hans. Neue Beiträge zur geometrischen Wertverteilungslehre. Comment. Math. Helv. **29**, 223–257 (1955).

The paper is concerned with the class of simply connected Riemann surfaces over the plane whose algebraic and logarithmic points project into a finite set of points. The problem is to find, as explicitly as possible, the order, the defects, and the ramification of the mapping function. The author adds greatly to the category of cases that can be treated. His results concern surfaces with a finite number of simply and doubly periodic ends. One important result is that there are no defects, in spite of the existence of logarithmic branch-points. The more refined results cannot be adequately described without reference to the figures. *L. Ahlfors* (Cambridge, Mass.).

Künzi, Hans P. Über periodische Enden mit mehrfach zusammenhängendem Existenzgebiet. Math. Z. **61**, 200–205 (1954).

Künzi, Hans P. Konstruktion Riemannscher Flächen mit vorgegebener Ordnung der erzeugenden Funktionen. Math. Ann. **128**, 471–474 (1955).

L'A. utilise la représentation de certaines surfaces de Riemann par leur "arbre topologique" de Speiser-Nevanlinna [voir G. Elfving, Acta Soc. Sci. Fenn. Nova Ser. A. **2**, no. 3 (1934)]. Dans le mémoire analysé ci-dessus il a

ainsi construit des surfaces de Riemann simplement connexes qui ont un nombre fini de "bouts" simplement et doublement périodiques; il les a représentées conformément sur un plan z, et a étudié la répartition des valeurs de la fonction génératrice, au point de vue de l'ordre, du défaut, et de l'indice de ramification. Dans le premier des deux mémoires ci-dessus, il étend les résultats au cas d'une surface à connexion finie, et obtient des formules relatives à l'ordre, au défaut et à l'indice, tels qu'ils ont été introduits dans le cas multiplement connexe par Ahlfors et Wittich [Wittich, Math. Ann. 122, 37–46 (1950); MR 12, 251].

Dans le deuxième mémoire, il construit une surface de Riemann parabolique dont la fonction génératrice est d'ordre égal à un nombre réel donné λ tel que $1<\lambda<\infty$. L'A. utilise les représentations quasi-conformes, et le "Verzerrungssatz" de Teichmüller et Wittich [Wittich, Math. Z. 51, 278–288 (1948); MR 10, 241] qui s'énonce ainsi: Soit une représentation biunivoque et quasi-conforme du plan z sur le plan w, et D le quotient de dilatation en un point. Si l'intégrale $\iint(D-1)|z|^{-2}dxdy$ converge, alors $|w/z|\to$constante quand $|z|\to\infty$. *R. de Possel.*

Rosenthal, Jenny E. Critical appraisal of the validity of standard techniques of conformal mapping. J. Washington Acad. Sci. 44, 276–280 (1954).

The author discusses the problem of mapping the strip $|\eta|<1$ of the $\zeta(=\xi+i\eta)$-plane onto the $z(=x+iy)$-plane, slit by the two semi-infinite lines $x<0$, $|y|=1$. The classical solution of Helmholtz $z=\zeta+\{\exp(\pi\zeta)+1\}\pi^{-1}$ and its application to the electrostatics of a semi-infinite plate condenser is discussed. A generalization of the mapping function in the form $z=\zeta+\sum C_n e^{n\pi\zeta}$ is proposed and conditions are given that this mapping carry the lines $\eta=\pm1$ in a one-to-one manner into the doubly-connected semi-infinite lines $x<0$, $y=\pm1$. The author does not discuss the fact that this mapping is not one-to-one in the interior of the infinite strip as follows from the uniqueness theorem of conformal mapping. Since the potential problem involves the inverse mapping function, the physical significance of the solution proposed would need some elaboration. *M. Schiffer.*

n order ***Lelong-Ferrand, Jacqueline. Représentation conforme et transformations à intégrale de Dirichlet bornée.** Gauthier-Villars, Paris, 1955. viii+259 pp. 4000 francs.

This book is a systematic exposition of those properties of conformal mapping which depend on the fact that a conformal mapping (into a bounded domain) has a finite Dirichlet integral. Thus, most of the book is concerned with mappings of a plane domain which are subject to only two conditions: 1) the Dirichlet integral of the mapping is finite and 2) the diameter of the image of a set is bounded in terms of the diameter of the image of its boundary. From these two properties alone, the author derives in Chapter III the Carathéodory theory of boundary correspondence and prime ends. Chapter IV is devoted to the study of sequences of mappings satisfying these conditions and much of the Carathéodory theory of domain convergence is derived.

Only in Chapters V and VI does the author restrict herself to conformal mappings. In Chapter V functions on a lattice are discussed and various relations are derived between preharmonic and preholomorphic functions on the one hand and harmonic and holomorphic functions on the other. Chapter VI gives an exposition of the length-area principle of Grötzsch and some of its applications. The final chapter deals with the problem of defining a class of functions with a finite Dirichlet integral in such a manner that limits of functions of this class are still in the class. A class of functions similar to those of Beppo Levi are introduced.

Although people working in elliptic differential equations are familiar with much of this material (some of it from the author's own papers), this book is the only place known to the reviewer where it is all conveniently and systematically assembled. This book is thus a very welcome addition to the literature in this field. *H. L. Royden.*

Robinson, Raphael M. Extremal problems for star mappings. Proc. Amer. Math. Soc. 6, 364–377 (1955).

The problem of maximizing

$$\Re\left\{\sum_1^n \beta_\nu\phi(z_\nu)+\psi\left(\sum_1^n \beta_\nu z_\nu\right)\right\}$$

for given $\phi(z)$ and $\psi(z)$ regular in $|z|\leq1$ and parameters satisfying

$$|z_\nu|=1,\quad \beta_\nu>0,\quad \sum_1^n\beta_\nu=1$$

is considered. The results are applied to show that the maximum possible value of

$$\Re\{B\log f'(z)+C\log[f(z)z^{-1}]\}$$

at a point z_0 ($|z_0|<1$) for the class of functions $f(z)=z+\cdots$ which map $|z|<1$ on star-shaped regions is attained when the region is the whole plane cut by one or by two radial slits. For certain values of B and C more precise results are obtained. For example, the set of possible values of $\log f'(re^{i\theta})$ is the map of $|z|\leq r$ by $\log(1+z)-3\log(1-z)$ at least if $r\leq0.6$. The results generalise work by Goodman, Marx, Stroganoff and the present author [see A. W. Goodman, same Proc. 4, 278–286 (1953); MR 14, 739].

A. J. Macintyre (Aberdeen).

***Royden, H. L. Conformal deformation.** Lectures on functions of a complex variable, pp. 309–313. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Deux représentations conformes f_0, f_1 d'un domaine Z sur un domaine W de même ordre de connexion sont dites conformément isotopes s'il existe une application continue $g(z,t)$ de $Z\times I$ dans W conforme pour chaque $t\in I$, telle que $g(z,0)=f_0(z)$ et $g(z,1)=f_1(z)$. Résultat principal: Deux représentations f_0, f_1, assujetties à faire correspondre un couple de points intérieurs donnés, sont conformément isotopes si elles sont homotopes. La démonstration s'appuie sur le lemme suivant: une représentation extrémale (i.e. non conformément isotope à une représentation de Z sur un domaine proprement contenu dans W) est la seule représentation conforme dans sa classe d'homotopie.

J. Lelong (Lille).

***Titus, C. J. The image of the boundary under a local homeomorphism.** Lectures on functions of a complex variable, pp. 433–435. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Soit ζ un courbe orientée du plan w: sous quelles conditions topologiques sur ζ existe-t-il une application $w=w(z)$ interieure sur D ($|z|<1$) continue sur $\bar{D}$ appliquant δ ($|z|=1$) sur ζ? L'auteur donne une condition globale sur la disposition des points doubles de ζ pour l'existence d'une telle application.

L. Fourès (Princeton, N. J.).

★Kaplan, Wilfred. Curve families and Riemann surfaces. Lectures on functions of a complex variable, pp. 425–432. The University of Michigan Press, Ann Arbor, 1955. $10.00.

A regular curve family is one which is locally homeomorphic to a family of parallel lines. The author considers regular curve families which fill an open, simply-connected domain of the plane and he gives a new proof of his theorem that every such family is homeomorphic to the level curves of a harmonic function $u(x, y)$ defined either on the unit disk or the entire xy-plane. The method of proof is to construct directly the Riemann surface of the inverse of an analytic function having $u(x, y)$ as real part by using the abstract order relations of the curves, i.e. the "normal chordal system." The proof depends only on the author's two early papers on the subject [Duke Math. J. 7, 154–185 (1940); 8, 11–46 (1941); MR 2, 322] and in particular does not use the fact that a family of the type considered is homeomorphic to a family defined by differential equations as did his original proof [Trans. Amer. Math. Soc. 63, 514–522 (1948); MR 9, 606], so that it is an immediate consequence of the present proof that every such curve family is homeomorphic to the family of solutions of a system of differential equations $\dot{x}=f(x, y)$, $\dot{y}=g(x, y)$, where f and g are of class C^∞ and $f^2+g^2\neq 0$. *W. M. Boothby.*

★Jenkins, James, and Morse, Marston. Conjugate nets on an open Riemann surface. Lectures on functions of a complex variable, pp. 123–185. The University of Michigan Press, Ann Arbor, 1955. $10.00.

A pair $[F, G)$ of families of curves on a surface Q is called a conjugate net if locally they are the respective level lines of the real and imaginary parts of an interior mapping. The question that concerns the authors is that of finding topological criteria on a conjugate net for the existence in the large of a conformal structure such that locally F and G are the real and imaginary level curves of an analytic function. The authors show that this is possible if and only if F and G are the level curves of an interior mapping of the universal covering surface of Q with simple transformation properties under cover transformations. In the latter sections explicit topological criteria are given for the special case that Q is the doubly-punctured sphere. *H.L. Royden.*

★Nevanlinna, Rolf. Countability of a Riemann surface. Lectures on functions of a complex variable, pp. 61–64. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Radó's theorem, which states that a Riemann surface has a countable basis, is established without the intervention of methods using the universal covering surface of a Riemann surface. The proof is achieved by the use of a metric constructed with the aid of a certain harmonic function whose existence is established by the classical alternating method of Schwarz [cf. the author's Uniformisierungstheorie, Springer, Berlin, 1953, pp. 145–148; MR 15, 208]. *M. H. Heins.*

★Nevanlinna, Rolf. Polygonal representation of Riemann surfaces. Lectures on functions of a complex variable, pp. 65–70. The University of Michigan Press, Ann Arbor, 1955. $10.00.

The author treats the type problem for doubly-connected regions obtained by analytic identification of two sides of a curvilinear triangle. Such a doubly-connected region is conformally equivalent to an annular domain: $1<|\omega|<\rho\leq\infty$. If $\rho<\infty$, it is termed hyperbolic, and if $\rho=\infty$, it is termed parabolic. Sufficient conditions are obtained for the parabolic case through the intervention of a C' map of the original curvilinear domain onto the infinite halfstrip: $\Re\, z\geq 0$, $0\leq\Im\, z\leq 1$. The sufficient conditions in question are expressed by the divergence of integrals defined with the aid of such auxiliary maps, the integrands having geometric interpretation in terms of the dilatation ratio. Special cases are considered. An example of the hyperbolic case is given. The availability of the methods of the present paper to establish the Ahlfors-Grötsch theorem is pointed out.

M. H. Heins (Providence, R. I.).

★Weyl, Hermann. Die Idee der Riemannschen Fläche. 3te Aufl. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1955. viii+162 pp. DM 22.00.

It does not occur frequently that a book which has been a classic for four decades appears in a completely rewritten edition. The event is the more significant as it concerns a work which has undoubtedly had a greater influence on the development of geometric function theory than any other publication since Riemann's dissertation.

The book originated in Weyl's lectures at Göttingen in the winter semester 1911–12. This course grew into the first edition of "Die Idee der Riemannschen Fläche" [Teubner, Leipzig-Berlin, 1913]. An appendix "Strenge Begründung der Charakteristikentheorie auf zweiseitigen Flächen" was added in the second edition [ibid., 1923], which together with its appendix also was reprinted in the United States [Chelsea, New York, 1947]. It was the author's desire to fuse the supplement with the main part of the book that led him to make a complete revision.

It seems appropriate to summarize here the historical background of the new book, as the earlier editions did not fall within the life span of Mathematical Reviews. The reader's interest is, of course, focused on the question: how did the subject of the book, the idea of abstract Riemann surface, come into being? The author's reference to Klein as the originator suggests a closer scrutiny, which turns out to be rewarding and not devoid of surprises.

It is well known that Riemann introduced the surfaces now bearing his name as covering surfaces of the complex plane. Although various passages of his other writings can be interpreted as suggesting a more abstract approach, there is nowhere an explicit mention of it. Klein's first thought in this direction was inspired by Prym. In the preface to the former's book, "Ueber Riemann's Theorie der algebraischen Functionen und ihrer Integrale" [Teubner, Leipzig, 1882] we find the following significant note: "Er (Prym) erzählte mir, dass die Riemannschen Flächen ursprünglich durchaus nicht notwendig mehrblättrige Flächen über der Ebene sind, dass man vielmehr auf beliebig gegebenen krummen Flächen ganz ebenso komplexe Funktionen des Ortes studieren kann, wie auf den Flächen über der Ebene". In 1923, Klein supplements this acknowledgment with an interesting remark [Ges. Math. Abh., vol. III, Springer, Berlin, p. 479], which seems to have remained quite unnoticed. He says that he had sent his book to Prym and the latter had, in a letter of April 8, 1882, answered "ausdrücklich" that, although he (Prym) did not exactly recall his conversation with Klein, he had certainly not referred to ordinary surfaces as carriers of complex functions. Thus we are faced with the unique situation that the less restrictive idea of Riemann surface is due to some unknown comment made by Prym and misunderstood by Klein.

The formal priority naturally remained with Klein, as he goes on to state: "Die Idee, geschlossene Flächen im Raume der funktionentheoretischer Betrachtung zugrunde zu legen . . . , scheint vor meiner Schrift nicht hervortreten zu sein. Sie hat auch wenig Nachahmung gefunden. Einzig H. Poincaré . . . und H. Weyl (1913, in seinem Buche, Die Idee der Riemannschen Fläche) scheinen eine Ausnahme zu machen."

But there still was a long way to go from Klein's idea to that of Weyl. First of all, the former only considered closed surfaces, which he covered "dachziegelartig" by overlapping planar regions. Furthermore, his reasoning was purely intuitive, based on physical illustrations, without any attempt at mathematical rigour. Weyl achieved perfection in both respects: he was the first to consider a Riemann surface in its full generality, and he gave the first axiomatic treatment of the concept.

The main advantage of the new notion was that it offered the natural carrier for analytic functions. Analyticity being locally defined, the globally imposed complex plane could only allow an artificially restricted theory of complex functions. Weyl's locally defined surface removed this obstacle and also permitted a full exploitation of Riemann's revolutionary idea of reversing the conventional process by giving the surface a priori and studying functions on it a posteriori.

Oddly enough, it took 35 years to find out whether the introduction of abstract surfaces had added any new specimens to conformal equivalence classes of covering surfaces of the plane. Only in 1948 Behnke and Stein [Math. Ann. **120**, 430–461 (1949); MR **10**, 696] succeeded in proving that this was not the case: there exists on every open abstract Riemann surface a nonconstant single-valued analytic function. Even more was then clear: the corresponding exponential function gives a conformally equivalent covering surface of the plane punctured at the origin. This result in turn is the best possible, the Picard theorem prohibiting two or more punctures.

For constructing functions on abstract surfaces Weyl used Dirichlet's principle. It is known that Riemann not only failed to give a rigorous proof of this principle but that he did not even make an effort to establish it beyond its intuitive physical meaning; Weierstrass's criticism made no impression on him [Klein, loc. cit., p. 493]. It is believed that Weierstrass left it to his student Schwarz to develop the linear method which then furnished the first strict proof of Riemann's results. Then at the turn of the century Hilbert succeeded in putting Dirichlet's principle on a sound basis, but his proof was still unnecessarily involved. Weyl introduced an essential simplification by his step-function with continuous derivatives. That this contribution, accepted in every text, is due to Weyl, does not seem to have been generally noticed. Dirichlet's principle has then firmly established itself into mathematical thinking, perhaps to the extent that one could regret the oblivion which for some seven decades was the fate of Schwarz's ingenious idea, capable of extensions far beyond its original scope.

In the theory of differentials, the fruitfulness of Weyl's approach is exemplified, as he now remarks, by the fact that the real harmonic rather than complex holomorphic linear differential forms have served as the starting point in recent generalizations to higher dimensions. In particular, the presentation in "die grosse Arbeit von Kunihiko Kodaira" [Ann. of Math. (2) **50**, 587–665 (1949); MR **11**, 108] reflects the pattern of the 1913 book.

Although Weyl restricted his attention to the theory of Riemann surfaces, the influence of his book has extended a great deal further. The work was written shortly after Brouwer's topological discoveries, and it was the author's ambition to give Riemann's theory the first presentation in which topological concepts and theorems received up-to-date rigorous treatment. The resulting axiomatic way of reasoning has inspired thinking in many doctrines quite remote from complex analysis.

How, then, does the new edition differ from the old one? Roughly speaking, the book is new in treatment while the subject matter has essentially remained unchanged. First of all, the abstract Riemann surface has undergone further development. It was originally defined as a two-dimensional manifold which has a conformal structure and can be triangulated. We owe to Radó [Acta Litt. Sci. Szeged **2**, 101–121 (1925)] the fact that the triangulability is equivalent to the existence of a countable basis for open sets and that the latter in turn is a consequence of the conformal structure. The triangulability axiom can thus be omitted. In the new book the Riemann surface appears as a neighborhood space with conformal structure.

Weyl, however, has come to this definition along another line of reasoning. He had used the combinatorial approach even more extensively than in the first edition, in his paper "Über kombinatorische und kontinuumsmässige Definition der Überschneidungszahl zweier geschlossener Kurven auf einer Fläche" [Z. Angew. Math. Phys. **4**, 471–492 (1953); MR **15**, 460]. But we now learn that he had subsequently preferred to define the intersection number by "topologizing" his method of constructing Abelian integrals (1913). This led him to drop the axiom of triangulation.

As a consequence of the new starting point, the proofs based on triangulation have been replaced by others using overlapping neighborhoods. This is the essential difference between the two editions. In the function-theoretic part the new treatment of the intersection number is conspicuous. Several notations have been changed. An example is the former "Überlagerungsfläche der Integralfunktionen" which now appears as "Klassenfläche".

The author has purposely avoided any extension of the book beyond its original scope of analyzing the concept of Riemann surface and developing function theory mainly on closed surfaces. Apart from the change of standpoint he discusses none of the new problems, methods, or results that have been developed in the theory of open Riemann surfaces since the appearance of the first edition. This is, in particular, true of Weyl's own method of orthogonal projection [Duke Math. J. **7**, 411–444 (1940); MR **2**, 202].

It remains to list the contents of the book, the first eleven sections of which are devoted to the concept and topology of Riemann surfaces, the remaining eleven, to functions on Riemann surfaces. (1) Weierstrass's concept of analytic function. (2) Concept of "analytisches Gebilde". (3) Mutual relation of the concepts of analytic function and "analytisches Gebilde". (4) Concept of two-dimensional manifold. (5) Examples of surfaces. (6) Specialization, in particular differentiable and Riemann surfaces. (7) Orientation. (8) Covering surfaces. (9) Differentials and line integrals. Homology. (10) Density and surface integrals. The residue theorem. (11) Intersection number. (12) Dirichlet's integral and harmonic differentials. (13) Construction of a potential flow with a double source. (14) The proof. (15) Elementary differentials. (16) Laws of symmetry. (17) The single-valued functions on F (Riemann surface) as a subclass of additive and multiplicative functions on $\hat{F}$ ("Klassenfläche"). Rie-

mann-Roch's theorem. (18) Abel's theorem. (19) The algebraic function field. (20) Uniformization. (21) Riemann surface and noneuclidian geometry. Fundamental regions. Poincaré's θ-series. (22) Conformal self-mapping of a Riemann surface.

The presentation throughout flows smoothly and beautifully. The absolute contrast with "assembly-line technique" is pleasantly accentuated by an interspersion of a humanistic element. A fascinating example is the motivation of the notation "verstreut" for an infinite point set without points of accumulation; the author tells us he is thinking of a poem by Hermann Hesse: "Sommernacht hat ihre dünnen Sterne verstreut, Jugendgedächtnis duftet im mondhellen Laub".

L. Sario (Los Angeles, Calif.).

Roşculeţ, Marcel N. Une théorie des fonctions d'une variable hypercomplexe dans l'espace à trois dimensions. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. 5, 361–401 (1954). (Romanian. Russian and French summaries)

The author considers a hypercomplex variable of the form $w=x+y\theta+z\theta^2$, where x, y, z are real and 1, θ, θ^2 are basis elements with $\theta^3=1$. The algebraic properties of w and a modulus $\|w\|$ are studied. Various analogs of ordinary complex variable theory are obtained such as a representation in exponential form, an analog of conformal mapping, conjugate functions, Cauchy-Riemann (Scheffers) equations, and integral theorems. The characteristics of the Scheffers equations are related to the divisors of zero of the algebra. [Further references to the literature of related and in some respects more general theories are given in E. Hille, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948, Chap. 5; MR 9, 594.]

P. W. Ketchum (Urbana, Ill.).

Roşculeţ, Marcel N. Fonctions d'une variable hypercomplexe dans l'espace à n dimensions. Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. 5, 135–145 (1953). (Romanian. Russian and French summaries)

Several results similar to those of the preceding paper are obtained for the more general case of a hypercomplex variable which is expressible as a linear combination of basis elements $1, \theta, \theta^2, \cdots, \theta^{n-1}$, where θ satisfies an algebraic equation of degree n with real coefficients.

P. W. Ketchum (Urbana, Ill.).

Roşculeţ, Marcel N. Fonctions d'une variable hypercomplexe dans l'espace à n dimensions. Fonctions conjuguées. Acad. Repub. Pop. Romåne. Bul. Şti. Secţ. Şti. Mat. Fiz. 5, 415–422 (1953). (Romanian. Russian and French summaries)

The discussion of the preceding paper is continued and additional integral theorems are obtained.

P. W. Ketchum (Urbana, Ill.).

Roşculeţ, Marcel M. Sur certaines équations aux dérivées partielles. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6, 489–498 (1954). (Romanian. Russian and French summaries)

The ideas of the preceding papers are used to obtain solutions of partial differential equations of the form

$$\Delta_1\Delta_2\cdots\Delta_k U=0,$$

where $\Delta_i=\sum a_{ij}\partial/\partial x_j$. The solutions are obtained in terms of arbitrary monogenic functions $f(w_1, w_2, \cdots, w_p)$ of the several independent variables $w_1, w_2, \cdots, w_p$, each of which is a hypercomplex variable of the type considered in the two preceding papers.

P. W. Ketchum (Urbana, Ill.).

Rinehart, R. F. The equivalence of definitions of a matric function. Amer. Math. Monthly 62, 395–414 (1955).

Various definitions have been given during the last 70 years of the value $f(A)$ where A is a square matrix and $f(z)$ a complex-valued function of the complex variable z. The author reports on eight relevant definitions and compares their relative merits and generality and discusses their mutual relationship. Since questions of priority are touched upon in the paper it might be mentioned that the definition by the Cauchy integral ascribed to Cartan (1928) has been proposed much earlier [S.-B. Preuss. Akad. Wiss. 1896, 7–16] by Frobenius. Careful consideration is given to the "laws of combination"; in particular: From $f(z)=g(z)\cdot h(z)$ follows $f(A)=g(A)\cdot h(A)$, and from $f(z)=g(h(z))$ follows $f(A)=g(h(A))$. There is also a brief discussion of the fact that for multiple-valued functions $f(z)$ there are values $f(A)$ which escape the usual (polynomial) definitions. In this connection a paper by von Neumann [Math. Z. 30, 3–42 (1929)] might have been quoted.

H. Schwerdtfeger (Melbourne).

Theory of Series

Šalát, Tibor. On sums of absolutely convergent series. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 4, 203–211 (1954). (Slovak. Russian summary)

Verf. betrachtet eine beliebige feste konvergente Reihe $\sum a_n$ mit positiven Gliedern, deren Summe A sei. Er stellt einige Eigenschaften der Menge W aller Summen $\sum \pm a_n$ fest. W ist eine perfekte Teilmenge des Intervalls $\langle -A, A\rangle$. Im Falle $a_k>R_k$ (Reihenreste)für alle k bestimmt Verf. die Komplementintervalle von W, im Falle $a_k\leqq R_k$ gilt sogar $W=\langle -A, A\rangle$.

K. Zeller (Tübingen).

Ruziewicz, S. Remarks on Dini's theorem about divergent series. Prace Mat. 1, 131–135 (1955). (Polish. Russian and English summaries)

Diese Note des verstorbenen Verfassers wurde von S. Hartmann druckfertig gemacht. In folgenden seien die s_n stets die Teilsummen einer (konvergenten oder divergenten) Reihe $\sum a_n$ mit positiven Gliedern. Satz 1. Die Reihe $\sum_n a_n/(s_n s^{\alpha}_{n-k})$ konvergiert, wenn k eine beliebige natürliche Zahl und $\alpha>0$ ist. [Der Fall $k=1$ stammt von Pringsheim; siehe K. Knopp, Theorie und Anwendung der unendlichen Reihen, 4. Aufl., Berlin-Heidelberg, 1947, S. 300; MR 10, 446.] Satz 2. Zu jedem Paar von Funktionen $\varphi(x)<1$ und $\psi(x)$ gibt es eine Reihe $\sum a_n$, so dass $\sum a_n/(s_n^{\varphi(n)} s_{n-1}^{\psi(n)})$ divergiert. [Beweis: Man setzt $s_1=1$, $s_n=\max(s_{n-1}^{\psi(n)/(1-\varphi(n))}, s_{n-1}+1)$.] Satz 3. Zu jeder Funktion $\varphi(x)\searrow 0$ (für $x\to\infty$) kann man eine Reihe $\sum a_n$ finden, für die $\sum a_n/s_n^{1+\varphi(n)}$ divergiert. [Beweis: Setze $s_n=1/\varphi(n)$.] Verf. fragt, ob die Reihe $\sum a_n/(s_n s_{[\alpha n]})$ bei $0<\alpha<1$ divergieren kann.

K. Zeller.

Chow, Hung Ching. A further note on the summability of a power series on its circle or convergence. Ann. Acad. Sinica. Taipei 1, 559–567 (1954).

Some results on the summability on $|z|=1$ of power series $f(z)=\sum_{n=0}^{\infty} c_n z^n$ converging in $|z|<1$ are given. We mention the following: Let $1\leqq p\leqq 2$ and $\alpha>(2/p)-1$, then $\int_0^1\int_0^{2\pi}|f'(re^{i\theta})|^p d\theta dr<\infty$ implies that $\sum c_n e^{in\theta}$ is absolutely summable (C, α) for almost all θ. The methods of proof are similar to those of an earlier paper [Proc. London Math. Soc. (3) 1, 206–216 (1951); MR 13, 340].

A. Dvoretzky (New York, N. Y.).

Chandrasekharan, K., and Minakshisundaram, S. A note on typical means. J. Indian Math. Soc. (N.S.) **18**, 107–114 (1954).

The authors desire to clarify some points in their book [Typical means, Oxford, 1952; MR **14**, 1077]. This became necessary in order to remove possible doubts as to the correctness of some statements in their book, after B. Kuttner's review of it [Bull. Amer. Math. Soc. **60**, 85–88 (1954)]. The review has suggested, for instance, some "shortcomings" concerning the specification of the degree of generality of the function $A(x)$ in the statement of Theorem 1.81. This, and other points, seem to be handled satisfactorily in the present note. The technicalities of the discussion do not allow us to give more details. *S. Mandelbrojt.*

Heywood, Philip. Integrability theorems for power series and Laplace transforms. J. London Math. Soc. **30**, 302–310 (1955).

The author proves that if $F(x)=\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence 1 and the c_n are ultimately positive, then for $\gamma<1$, $(1-x)^{-\gamma}F(x)$ is $L(0, 1)$ if and only if $\sum n^{\gamma-1}c_n$ converges: if $\sum_{n=0}^{\infty} c_n=0$ then the same is true for $1<\gamma<2$, and $F(x)/(1-x)$ is $L(0, 1)$ if and only if $\sum c_n \log n$ converges. Analogous results are obtained for Laplace transforms: for example, if $\phi(t)$ is positive for $t>0$ and $e^{-st}\phi(t)$ is $L(0, \infty)$ for $s>0$, with $\Phi(s)=\int_0^\infty e^{-st}\phi(t)dt$, then (for $\gamma<1$) $s^{-\gamma}\Phi(s)$ is $L(0, 1)$ or $L(1, \infty)$ if and only if $t^{\gamma-1}\phi(t)$ is respectively $L(1, \infty)$ or $L(0, 1)$. *R. P. Boas, Jr.*

Bajšanski, B., and Bojanić, R. On two sums. Bull. Soc. Math. Phys. Macédoine **5** (1954), 34–36 (1955). (Serbo-Croatian. English summary)

Simple proofs of the formulae

$$\sum_{k=0}^{n-1}(n-k)\frac{(2n+2k+1)!!}{(2n+2k)!!}\,\frac{(2n-2k-1)!!}{(2n-2k)!!}=\frac{n(2n+1)}{4},$$

$$\sum_{k=1}^{n}k^2\frac{(2n+2k-1)!!}{(2n+2k)!!}\,\frac{(2n-2k-1)!!}{(2n-2k)!!}=\frac{n(2n+1)}{8},$$

where $(2k)!!=2\cdot4\cdots(2k)$, $(2k+1)!!=1\cdot3\cdots(2k+1)$, are given. These formulae appear in a paper of Ê. L. Bloh [Prikl. Mat. Meh. **17**, 705–726 (1953); MR **16**, 82]. He verified them for $n=1, 2, 3, 4, 5$ and when $n\to\infty$ and supposed therefore that they are true for every integral n, But a rigorous proof of these assertions he could not find.

It may be observed that more general formulae of the same kind may be obtained by considering the generating functions of Legendre or Gegenbauer's polynomials. (Authors' English summary.) *R. Finn.*

Fourier Series and Generalizations, Integral Transforms

★**Natanson, I. P. Konstruktive Funktionentheorie.** Akademie-Verlag, Berlin, 1955. xiv+515 pp. DM 36.00.

Translation by K. Bögel of the author's Konstruktivnaya teoriya funkciĭ [Gostehizdat, Moscow, 1949; MR **11**, 591].

Kinukawa, Masakiti. On the Cesàro summability of Fourier series. Tôhoku Math. J. (2) **6**, 109–120 (1954).

Modifying W. H. Young's criterion for the convergence of a Fourier series, G. H. Hardy and J. E. Littlewood [Ann. Scuola Norm. Sup. Pisa (2) **3**, 43–62 (1934)] proved that if

$$\text{(i)}\qquad \int_0^t |\varphi(u)|\,du=o(t/\log t^{-1})$$

and

$$\text{(ii)}\qquad \int_0^t |d\{u^\Delta\varphi(u)\}|=O(t)$$

for small t, where $\Delta\geqq1$, then the Fourier series of the even function $\varphi(t)$ is convergent at the origin. G. Sunouchi [Tôhoku Mat. J. (2) **3**, 216–219 (1951); **4**, 187–193 (1952); MR **13**, 739; **14**, 552] modified the hypotheses by replacing one or both of (i) and (ii) by one or both of

$$\text{(i}'\text{)}\qquad \int_0^t \varphi(u)du=o(t^\Delta)$$

and

$$\text{(ii}'\text{)}\qquad \lim_{k\to\infty}\limsup_{x\to+0}\int_{(xk)^{1/\Delta}}^{\pi}\frac{\varphi(t+x)-\varphi(t)}{t}du=0,$$

respectively, the cases $\Delta=1$ giving results of S. Pollard and J. J. Gergen [see Gergen, Quart. J. Math. Oxford Ser. **1**, 252–275 (1930)]. Here the author modifies these various hypotheses to obtain conditions for the summability (C, ρ), $-1<\rho<1$, of the Fourier series. In (i) he inserts a factor $t^{-\sigma}$, $\sigma=\rho(\Delta-1)(1+\rho)^{-1}$, on the right hand side; in (i′) he inserts $t^{-2\sigma}$ similarly; in (ii′) he inserts a factor $(x/t)^\rho$ in the integrand and replaces the difference by a higher difference, after the manner of Gergen, loc. cit. *L. S. Bosanquet.*

Kinukawa, Masakichi. On the strong summability of the derived Fourier series. Proc. Japan Acad. **30**, 801–804 (1954).

The author attempts to extend theorems of B. N. Prasad and U. N. Singh [Math. Z. **56**, 280–288 (1952); **57**, 481–482 (1953); MR **14**, 370] from strong summability to strong summability with power $k>0$. His proofs are incomplete since he neglects the contributions from the range (δ, π), $\delta>0$. *L. S. Bosanquet* (London).

Sunouchi, Gen-ichiro. Notes on Fourier analysis. XLVII. Convexity theorems and Cesàro summability of Fourier series. J. Math., Tokyo **1**, 104–109 (1953).

It is familiar that results concerning the convergence or summability of a Fourier series may be obtained by applying successively a quasi-Tauberian theorem and a Tauberian theorem [for the term quasi-Tauberian see N. Wiener, Ann. of Math. (2) **33**, 1–100 (1932), Ch. VII]. For example, F. T. Wang proved [Sci. Rep. Tôhoku Imp. Univ. Ser. 1. **24**, 697–700 (1936); Proc. London Math. Soc. (2) **47**, 308–325 (1942); MR **4**, 37] the Hardy-Littlewood theorem that

$$\text{(i)}\qquad a_n>-Kn^{-\delta},\quad 0<\delta<1,$$

and

$$\text{(ii)}\qquad \int_0^t |\varphi(u)|\,du=o(t/\log t^{-1})$$

imply the convergence of the Fourier series, by showing first that (ii) implies the summability by a certain Riesz mean and then combining this result with the Tauberian condition (i). Here the author generates other theorems of this type, using a convexity theorem for the second step. For example, if (i) $a_n>-Kn^{-(1-\delta)}$, $0<\delta\leqq1$, and $\phi_\beta(t)=o(t^y)$, $y>\beta\geqq0$, where $\phi_\beta(t)$ is a βth functional integral, then the Fourier series is summable (C, σ) for $\sigma>(\beta-y+\delta y)(y+\delta-\beta)^{-1}$

[presumably provided $\rho \geqq 0$]. The author points out that in certain cases it is already known that the conclusion holds with $=$ in place of $>$, and suggests that a similar improvement may be possible in all his theorems.

L. S. Bosanquet (London).

Kinukawa, Masakiti. On the integro-jump of a function and its Fourier coefficients. Proc. Japan Acad. 31, 45–48 (1955).

Let $f(x)$ be periodic of period 2π and Lebesgue integrable in $(-\pi, \pi)$, and let $\psi(x)=f(x+t)-f(x-t)-L(x)$ for some number $L(x)$. Let $t_n^\alpha(x)$ be the nth Cesàro mean of order α of the conjugate Fourier series of $f(x)$. If (*) $\int_0^t \psi(u)du=o(t^{1/\alpha})$ as $t \to 0$ for some $0<\alpha<1$. Then

$$\lim_{n\to\infty} \{t^\alpha{}_{2n}(x)-t_n^\alpha(x)\}=L(x)\pi^{-1}\log 2.$$

Another consequence of (*) is that

$$\lim_{n\to\infty} n(b_n \cos nx-a_n \sin nx)=-L(x)/\pi \quad (C, 1+\alpha).$$

P. Civin (Eugene, Ore.).

Tureckiĭ, A. H. On best approximation of periodic differentiable functions in the space L. Dokl. Akad. Nauk SSSR (N.S.) 101, 1001–1004 (1955). (Russian)

The author considers a (Čebyšev) system $\{\phi\}$ of continuous functions of period 2π such that any nontrivial linear combination of $(\phi_1, \cdots, \phi_n)$ has at most $n-1$ zeros on $0 \leqq x<2\pi$. Let $W^{(r)}$ be the class of functions of period 2π with mean value 0 and an rth derivative satisfying $|f^{(r)}(t)| \leqq 1$. He calculates the minimum over all sets $\{\phi\}$ of the least upper bound for $f \in W^{(r)}$ of $\inf \int_0^{2\pi}|f(t)-\phi(t)|dt$, where $\phi(t)$ is a linear combination of the first n of the ϕ_k and the inf is taken over all $\phi(t)$. It is equal to

$$\frac{16}{\pi}\left(\frac{2}{n+1}\right)^r \sum_{q=0}^{\infty} \frac{(-1)^{qr}}{(2q+1)^{r+2}},$$

and is attained for $\{\phi\}=\{1, \cos \nu x, \sin \nu x\}$, $1 \leqq \nu \leqq \frac{1}{2}(n-1)$.

R. P. Boas, Jr. (Evanston, Ill.).

Stečkin, S. B. On absolute convergence of orthogonal series. Dokl. Akad. Nauk SSSR (N.S.) 102, 37–40 (1955). (Russian)

Let $\{\phi_n\}$ be an orthonormal set which is complete in L^2, and let $f=\sum c_n\phi_n$ be the expansion of a given f. The author has given a sufficient condition for the convergence of $\sum|c_n|$ [Mat. Sb. N.S. **29**(71), 225–232 (1951); Amer. Math. Soc. Transl. no. **89** (1953); MR **13**, 229; **15**, 28]. He now modifies this and obtains a condition which is necessary and sufficient, thus in a sense solving the absolute convergence problem. Consider all possible n-term linear combinations $\Phi_{(n)}=\sum_{i=1}^n d_{k_i}\phi_{k_i}$, $k_1<k_2<\cdots$, and put $e_n(f)=\inf \|f-\Phi_{(n-1)}\|$ (L^2 norm). Thus $e_n(f)$ measures the best L^2 approximation to f by linear combinations of $n-1$ functions ϕ_k, chosen without regard to their order of occurrence in the set $\{\phi_n\}$. Then $\sum|c_n|$ converges if and only if $\sum n^{-\frac{1}{2}}e_n(f)$ converges. The author also states two theorems on trigonometric Fourier series. Let $f(x) \in L^2$ and have complex Fourier coefficients c_n. If $|c_n|$ decreases then $\sum|c_n|$ converges if and only if $R=\sum n^{-\frac{1}{2}}\omega_2(1/n)$ converges, where $\omega_2(\delta)$ is the L^2 modulus of continuity of f; the series $\sum|c_n|$ and R are simultaneously convergent or divergent if and only if there is a sequence of numbers $\mu_n \uparrow \infty$ such that $\sum\mu_n^{-2}<\infty$ and $\sum(\mu_n|c_n|)^2<\infty$.

R. P. Boas, Jr. (Evanston, Ill.).

Riesz, Marcel, and Livingston, A. E. A short proof of a classical theorem in the theory of Fourier integrals. Amer. Math. Monthly 62, 434–437 (1955).

Pringsheim's theorem that the Fourier integral formula holds for a function of bounded variation tending to 0 at infinity is proved: the proof uses a new inequality for the integral of the product fg, g continuous, f of bounded variation.

J. L. B. Cooper (Cardiff).

Yoshihiro, Takashi. Some theorems of Fourier integral. J. Math., Tokyo 1, 87–93 (1953).

Generalisations of the theorem of Hartman and Wintner [Proc. Amer. Math. Soc. **2**, 398–400 (1951); MR **13**, 30] that if (1) $f(t) \downarrow 0$ as $t \uparrow \infty$ and (2) $\int_0^\infty tf(t)dt=s$, then $F(t)=\int_0^\infty f(x) \sin xt\, dt$ exists and $\lim_{t\to 0} F(t)/t=s$. The first theorem, attributed to S. Izumi, states that the conclusion holds if (1) is replaced by $f(t)>-1/t^2$; and that if it is replaced by the conditions that the integrals $\int_u^{2u} (x|f(x)|-xf(x))\, dx$ and $\int_0^u xf(x)\, dx$ are bounded then $F(t)/t=O(1)$ as $t\to 0$.

Further theorems concern integrals of the form

$$F(t)=\int_0^\infty f(t)\varphi(x, t)dt.$$

Firstly, if $f(t)>-1/t^2$ and $\int_0^\infty tf(t)dt=s$, and if

$$\varphi(x, t)=\varphi(xt),$$

where $\varphi(t)$ is a bounded measurable function on $(0, \infty)$ with period e and such that $(0, e)$ can be divided into a finite number of intervals in each of which $\varphi(t)/t$ is monotonic, and such that for constants φ_0, A, $|t^{-1}\varphi(t)-\varphi_0| \leqq At$ as $t\to 0$, then $F(t)$ exists, and $\lim_{t\to 0} F(t)/t=\varphi_0 S$. Secondly, if $f(t)$ is non-negative, $\varphi(x, t)$ is a bounded measurable function for $0 \leqq x \leqq \infty$, $0 \leqq t \leqq 1$, and there is a function ψ such that $\int_0^\infty \psi(t)f(t)dt=S$, $|\varphi(x, t)/t\psi(t)-1| \leqq Axt$, and $|\varphi(x, t)/t\psi(x)| \leqq A$ for a constant A, then $F(t)$ exists and $\lim_{t\to 0} F(t)/t=s$.

J. L. B. Cooper (Cardiff).

Methée, Pierre-Denis. Transformées de Fourier de distributions invariantes. C. R. Acad. Sci. Paris 240, 1179–1181 (1955).

Cette note est un complément à la Thèse de l'auteur [Comment. Math. Helv. **28**, 225–269 (1954); MR **16**, 255] sur les distributions dans R^n qui sont invariantes par le groupe des rotations propres de Lorentz. L'auteur indique maintenant une méthode permettant de déterminer les images de Fourier de certaines de ces distributions, en particulier de celles qui interviennent dans la résolution de l'équation des ondes $(\square+k)T=0$. Il commence par observer que l'ensemble Ω des opérateurs ω proportionnels aux opérateurs invariants u, $\square$, $\nabla-h$, avec

$$u=x_n^2-\sum_{i=1}^{n-1}x_i^2, \quad \square=\frac{\partial^2}{\partial x_n^2}-\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}, \quad \nabla=\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i}$$

(h, constante quelconque),

est appliqué dans lui-même par la transformation $\mathfrak{F}$ de Fourier, c'est-à-dire: $\omega'(\mathfrak{F}T)=\mathfrak{F}(\omega T)$ implique $\omega' \in \Omega$. Dans le cas $n=4$, on retrouve, sous le point de vue des distributions, les images de certaines "fonctions singulières" des physiciens, solutions de l'équation des ondes:

$$\mathfrak{F}\Delta=2i\pi\epsilon(\tau)\delta(k-u), \quad \mathfrak{F}\Delta^{(1)}=2\pi\delta(k-u), \quad \mathfrak{F}\bar{\Delta}=(k-u)^{-1}.$$

J. Sebastião e Silva (Lisbonne).

order ✱**Arens, R. F., and Calderón, A. P. Analytic functions of Fourier transforms.** Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 39–52. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954.

There is a well known result of Cameron and Wiener [Trans. Amer. Math. Soc. **46**, 97–109 (1939); MR **1**, 13] about the effect of compounding a Fourier-Stieltjes transform on the real line with an analytic function, which result has since been established by treating the class of Fourier-Stieltjes transforms as a Banach algebra and applying the methods initiated by Gelfand and his associates. The latter method of approach is used in the present paper to establish a two-fold extension of the Wiener-Cameron result. In the first place, the real line is replaced by a σ-compact, locally compact, abelian group G. In the second place, what is more interesting, the single-valued analytic function on a plane domain is replaced by a many-valued analytic function of several complex variables.

Denote by R_n the set of power-series elements in n complex variables having positive radius of convergence (the set of holomorphic function-germs on C^n); R_n is a (not connected) complex-analytic n-manifold which is a covering space of C^n with projection map π taking a power-series element into its centre. On R_n there is a distinguished (single-valued) holomorphic function $\mathfrak{F}$ assigning to each power series element the value of its constant term. The main theorem runs as follows: Let $f_1, \cdots, f_n$ be Fourier transforms of bounded measures on the dual of G. Suppose φ is a continuous function mapping G into R_n such that $\pi\varphi(s)=(f_1(s), \cdots, f_n(s))$ for s in G. Then there exists f_0, likewise the Fourier transform of some bounded measure, such that $f_0(s)=\mathfrak{F}[\varphi(s)]$ for s in G. A more concrete form of this theorem is given, wherein the "section" φ is defined implicitly by a suitably restricted holomorphic function on some domain in C^{n+1}.

Some general remarks on the "lifting up" (relèvement) of almost periodic functions are combined with the main theorem to yield results about the almost periodicity of functions which are algebraically dependent on given almost periodic functions, thereby generalising a theorem of Walther [Monatsh. Math. Phys. **40**, 444–457 (1933)].

R. E. Edwards (London).

Džrbašyan, M. M. On a new integral transform and its application in the theory of entire functions. Izv. Akad. Nauk SSSR. Ser. Mat. **19**, 133–190 (1955). (Russian)

Džrbašyan, M. M. On Abel summability of generalized integral transforms. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **7**(1954), no. 6, 1–26 (1955). (Russian. Armenian summary)

The author is devoting a series of papers to various aspects of the theory of generalized Fourier transforms having kernels $\exp((-zt)^\rho]$ and $E_\rho(zt, \mu)$, where $E_\rho(z, \mu)$ is the Mittag-Leffler function $\sum_{n=0}^\infty z^n/\Gamma(\mu+n/\rho)$. [Mat. Sb. N.S. **33**(75), 485–530 (1953); Izv. Akad. Nauk SSSR. Ser. Mat. **18**, 427–448 (1954); MR **15**, 517; **16**, 468]. The transforms are suitable for representing functions defined along one or more rays in the complex plane and in particular allow the author to generalize the Paley-Wiener representation of entire functions of exponential type to functions of order different from 1. The first paper under review deals in detail with the L^2 theory [cf. Dokl. Akad. Nauk SSSR (N.S.) **95**, 1133–1136 (1954); MR **15**, 947] and with the representation of entire functions for which either (A) f is of order ρ, $\frac{1}{2}\leqq\rho<1$, and

$$\int_0^\infty |f(te^{\pm\frac{1}{2}i\pi/\rho})|^2 t^{2\mu\rho-\rho-1}dt<\infty \quad (\tfrac{1}{2}<\mu<\tfrac{1}{2}+\rho^{-1});$$

(B) f is of order $\rho\geqq\frac{1}{2}$, mean type, and

$$\int_0^\infty |f(te^{i\alpha})|^2 t^{2\mu\rho-\rho-1}dt<\infty \quad (\tfrac{1}{2}<\mu<\tfrac{1}{2}+\rho^{-1})$$

for $\frac{1}{2}\pi/\rho\leqq\alpha\leqq2\pi-\frac{1}{2}\pi/\rho$; (C) f is of order ρ, $1\leqq\rho<2$, mean type, and

$$\int_0^\infty |f(t\exp\{\pm\tfrac{1}{2}\pi(1\pm\rho^{-1})i\})|^2 t^{2\mu\rho-\rho-1}dt<\infty \qquad (\tfrac{1}{2}<\mu<\tfrac{1}{2}+\rho^{-1});$$

or (D) f is of integral order $p\geqq1$, mean type, and

$$\int_0^\infty |f(te^{-i\pi(\frac{1}{2}+(n+\frac{1}{2})/p)})|^2 t^{\mu p-p-1}dt<\infty \qquad (\tfrac{1}{2}<\mu<\tfrac{1}{2}+p^{-1};\ n=0, 1, \cdots, 2p^{-1}).$$

There is a detailed study of the Mittag-Leffler functions which may be of independent interest.

The second paper establishes Abel summability of some generalized Fourier transforms at points where appropriate continuity conditions are satisfied. *R. P. Boas, Jr.*

Devinatz, A. The representation of functions as Laplace-Stieltjes integrals. Duke Math. J. **22**, 185–191 (1955).

The theory of reproducing kernels [see N. Aronszajn, Trans. Amer. Math. Soc. **68**, 337–404 (1950); MR **14**, 479; A. Devinatz, ibid. **74**, 56–77 (1953); **77**, 455–480 (1954); MR **14**, 659; **16**, 584] is used to prove necessary and sufficient conditions that a continuous function $f(x)$ of the 2-vector $x=(x_1, x_2)$ should have a representation $f(x)=\int_a^b e^{x\cdot t}d\alpha(t)$ with $\alpha(t)$ a non-negative measure, when x ranges over a convex plane set Q containing the origin .The conditions are that, for $x,y \in Q/2$, $f(x+y)\gg0$, and

$$a_k f(x+y)\ll\frac{f(x+y)}{x}\ll b_k f(x+y) \quad (k=1, 2).$$

Here the sign $f(x, y)\gg g(x, y)$ is used in the sense of Aronszajn to mean that $f(x, y)-g(x, y)$ is a positive definite function. *J. L. B. Cooper* (Cardiff).

Rooney, P. G. An application of some spaces of Lorentz. Canad. J. Math. **7**, 314–321 (1955).

The spaces $\Lambda(\alpha)$, $M(\alpha)$ were defined by Lorentz [Ann. of Math. (2) **51**, 37–55 (1950); MR **11**, 442]. In this paper necessary and sufficient conditions are given that a function $f(s)$ defined for $[0, \infty)$ be the Laplace transform of a function in $\Lambda(\alpha)$ and in $M(\alpha)$ where the domain of the functions in each case is $[0, \infty)$. In both cases one of the conditions is that $f^k(s)$ is defined for $k=1, 2, 3, \cdots$ and $\lim_{k\to\infty} f^k(s)=0$. The other conditions in both cases is that the norm $\|L_{k,}[f(s)]\|$ is uniformly bounded in k in the $\Lambda(\alpha)$ and $M(\alpha)$ norms respectively, where $L_k[f(s)]$ denotes the Widder-Post inversion operator [D. V. Widder, The Laplace transform, Princeton, 1941; MR **3**, 232].

R. E. Fullerton (College Park, Md.).

Griffith, J. L. A theorem concerning the asymptotic behaviour of Hankel transforms. J. Proc. Roy. Soc. New South Wales **88**, 61–65 (1955).

Let $\bar{f}(u)$ be the Hankel transform of order ν of $f(x)$. Assume that $\frac{1}{2}<a<2+\nu$, $x^{\frac{1}{2}}f(x)\to0$ as $x\to\infty$, $F(x)\equiv x^a f(x)$

is absolutely continuous in $0 \leqq x < \infty$, and $x^{\frac{1}{2}-a}F'(x)$ is absolutely integrable in $0 < \eta \leqq x \leqq \infty$.

It is shown that if $\lim_{x\to+0}\{x^a f(x)\} = K$, then

$$\lim_{u\to\infty} u^{2-a} f(u) = K\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}a+1)}{2^{a-1}\Gamma(\frac{1}{2}\nu+\frac{1}{2}a)}.$$

A finite number of simple discontinuities in $f(x)$ can be allowed if $\nu > -\frac{1}{2}$. *F. Goodspeed* (Vancouver, B. C.).

Kumar Jain, Mahendra. On Meijer transform. Acta Math. **93**, 121–168 (1955).

The author calls

$$\phi(s) = s\int_0^\infty e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}}W_{k,m}(st)f(t)dt$$

the Meijer transform of $f(t)$. In section I he computes a large number of Meijer transform pairs, all of which are particular cases of an integral evaluated by Ragab [Proc. Glasgow Math. Assoc. 1, 192–195 (1953), eq. (11); MR **15**, 424] and others. In section II he utilizes recurrence relations etc. of confluent hypergeometric functions to write down the corresponding relations for Meijer transforms, giving several examples. In section III the author points out that for two Meijer transform pairs the relation

$$\int_0^\infty f_1(t)\phi_2(t)t^{-1}dt = \int_0^\infty f_2(t)\phi_1(t)t^{-1}dt$$

holds, and gives numerous examples, most, if not all, of which are particular cases of an integral evaluated by C. S. Meijer [Nederl. Akad. Wetensch., Proc. **44**, 81–92 (1941); MR **2**, 287]. In section IV some relations involving Meijer transforms of functions self-reciprocal in the Hankel transformation are obtained.

A. Erdélyi (Pasadena, Calif.).

Chakravarty, N. K. On symbolic calculus of two variables. Acta Math. **93**, 1–14 (1955).

The principal "theorem" is: If $f(p)$ is the operational image of $x^{\nu-1}h(x)$, and $p^{\mu-\lambda}h(p^{-\mu})$ is the operational image of $\sigma(x)$, then the operational double-image of $y^{\nu-\lambda/\mu}\sigma(xy^{1/\mu})$ is $p^{\mu\nu-\lambda}f(p^{\mu}q)$. [The author's proof is somewhat involved but the result is obtained simply upon evaluating the Laplace double integral as an iterated integral.] Numerous examples of this result are stated.

A. Erdélyi (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Karmazina, L. N. Certain properties of the roots of Jacobi polynomials. Vyčisl. Mat. Vyčisl. Tehn. **2**, 108–110 (1955). (Russian)

$G_n(p, q, x)$, $n = 0, 1, \cdots$, being the orthogonal polynomials belonging to the weight function $x^{q-1}(1-x)^{p-q}$ on the interval $(0, 1)$, the author proves that $G_n(p, q, x)$ and $G_n(p+1, q, x)$ have no common zeros; that at a zero of $G_n(p, q, x)$, the polynomials $G_n(p-1, q, x)$ and $G_n(p+1, q, x)$ have opposite signs; that given ϵ $(0<\epsilon<1)$, n, and $p[q]$, it is possible to choose $q[p]$ so that G_n has a zero in $(0, \epsilon)$ or $(1-\epsilon, 1)$; that given n, x, and a sufficiently large p, it is possible to choose q so that $G_n = 0$. *A. Erdélyi* (Pasadena, Calif.).

Denisyuk, I. N. On a generalization of Laguerre polynomials and a Cauchy problem for partial difference equations connected with them. Ukrain. Mat. Ž. **6**, 245–256 (1954). (Russian)

The author defines two polynomials of degree n, $L_n(2t, \lambda)$ and $\Lambda_n(2t, \lambda)$, by writing the inverse Laplace transform of

$$\frac{(p-1)^n(1-p/\lambda)^n}{(p+1)^{n+1}(1+p/\lambda)^{n+1}}$$

in the form

$$e^{-t}L_n(2t, \lambda) + e^{-\lambda t}\Lambda_n(2t, \lambda).$$

[$L_n(2t, \infty) = L_n(2t)$ is Laguerre's polynomial.] The paper contains explicit expressions for some of the coefficients of these polynomials, recurrence relations for the others, differentiation formulas for the polynomials, and the like.

A. Erdélyi (Pasadena, Calif.).

Denisyuk, I. N. On polynomials of the problem of extensional impact. Ukrain. Mat. Ž. **6**, 423–429 (1954). (Russian)

Polynomials $M_n(x)$ were defined in an earlier paper [Dopovidi Akad. Nauk Ukrain. RSR **1954**, 79–81; MR **16**, 694]. In the present paper new polynomials are added: $\mathfrak{M}_n(2t)$ is the factor of e^{-t} in the second repeated integral of $e^{-t}L_n(2t)$. The paper contains several relations for these polynomials, a table of coefficients of $M_n(t)$ for $1 \leqq n \leqq 20$, the expansion of $\mathfrak{M}_n$ in a series of L_n or M_n, and the expansion of L_n and M_n in a series of $\mathfrak{M}_n$, and some integrals.

A. Erdélyi (Pasadena, Calif.).

✱González, M. O. Some analytical applications of developments in series of Legendre polynomials. Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 163–171. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish) on order

Let $f(x) = \sum_{n=1}^\infty A_n P_n(x)$, where $P_n(x)$ is the Legendre polynomial of degree n. The author determines the expression of A_n, when $f(x)$ is a Legendre function of the first kind, or an elliptic integral. This leads to a new, short proof of the formula

$$P_m(x) = \frac{\sin m\pi}{\pi}\sum_{n=0}^\infty(-1)^n\frac{2n+1}{(m-n)(m+n+1)}P_n(x)$$

for Legendre functions of non-integral order m [see Magnus and Oberhettinger, Formeln und Sätze für die speziellen Funktionen der mathematischen Physik, Springer, Berlin, 1943; MR **9**, 183] and to the following, apparently new expansions: Set $\lambda = k'^2 - k^2$; then

$$\mathrm{sn}^{-1}\, x = 2\sum_{n=0}^\infty\frac{(-1)^n}{2n+1}P_n(\lambda)v^{2n+1},$$

where v is defined by $x = 2v/(1+v^2)$;

$$\mathrm{cn}^{-1}\, x = 2\sum_{n=0}^\infty\frac{(-1)^n}{2n+1}P_n(\lambda)w^{2n+1},$$

with $w = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$; finally,

$$\wp^{-1}(x) = \sum_{n=0}^\infty\frac{(-1)^n}{2n+1}\gamma^{2n}P_n(\lambda)(x-e_1)^{-(n+\frac{1}{2})},$$

with $\gamma^2=3e_1/2\lambda$. For the complete elliptic integrals, one obtains

$$K=2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}P_n(\lambda),$$

$$E=4\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)}P_n(\lambda)$$

and similar formulae for K' and E'. The proofs are based on classical formulae from the theory of hypergeometric functions. *E. Grosswald* (Philadelphia, Pa.).

***Evgrafov, M. A. Interpolyacionnaya zadača Abelya-Gončarova. [The Abel-Gončarov interpolation problem.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 126 pp. 2.90 rubles.

The Abel interpolation series for $F(x)$ is a series of polynomials of degree n with coefficients $F^{(n)}(n)$. Gončarov introduced the generalization where the coefficients are $F^{(n)}(\lambda_n)$; the polynomials are then known as Gončarov polynomials. This monograph surveys various problems connected with these interpolation series, both expansion theorems and uniqueness theorems, with emphasis on recent Russian work; results of foreigners are mentioned, with references, but not covered in any detail.

The first chapter discusses direct estimates for Gončarov polynomials and the corresponding representation theorems. The second chapter deals with Gelfond's generalization of the problem: if $F(z)$ is an entire function of exponential type and $F(z)=(2\pi i)^{-1}\int_C f(w)e^{zw}dw$ is its Pólya representation, the functionals $F^{(n)}(n)$ are replaced by

$$L_n(f)=(2\pi i)^{-1}\int_C[u(w)]^n f(w)dw,$$

with $u(w)$ regular at 0, $u(0)=0$, $u'(0)\neq 0$. A further generalization is made by defining $L_n(F)$ for any entire F as $\sum_{k=0}^{\infty}a_{n,k}F^{(n+k)}(0)$ with appropriate restrictions on the coefficients. This generalized problem is treated by using a generalization of the Laplace-Borel transform, with an entire function $\Phi(z)$ with decreasing coefficients replacing e^z. The third chapter is devoted to detailed estimates of $\Phi(z)$ when various hypotheses are imposed. Chapter 4 treats the interpolation problem by applying general expansion theorems obtained by using infinite systems of linear equations. [One misses a reference to P. Davis, Duke Math. J. **20**, 345–357 (1953); MR **15**, 207. The author's theorem on p. 76 was proved by other methods by the reviewer, Trans. Amer. Math. Soc. **48**, 467–487 (1940), p. 477; MR **2**, 80; the author has apparently seen only an abstract which stated a weaker result.]

In Chapter 5 the function-theory apparatus of Chapters 2 and 3 is used to get several convergence theorems for the Gončarov problem. The central idea is that the interpolation problem can be replaced, via the generalized Borel transform, by the problem of expanding a function in terms of $z^n\Phi^{(n)}(\lambda_n z)$. The coefficients of the expansion of z^k satisfy recurrence relations which can be exploited to get useful estimates. A simple specimen of the results obtained here is that if $\lambda_{n+1}-\lambda_n\to 1$ the interpolation series converges for every entire function of exponential type whose indicator diagram is inside the region defined by $|we^w|<e^{-1}$. Chapter 6 discusses the special uniqueness problem of the minimum type for an entire function of exponential type (not identically zero) such that the function and every derivative have at least one zero apiece in the unit circle. [The problem was apparently first discussed by S. Bernstein in 1930; it was considered at about the same time by Takenaka, and the recent developments were inspired by the discussion in Whittaker's Interpolatory function theory, Cambridge, 1935.] The author discusses a generalization where $F^{(n)}(z)$ has $\nu(n)$ zeros in the circle. *R. P. Boas, Jr.*

Pollard, Harry. The Bernstein approximation problem. Proc. Amer. Math. Soc. **6**, 402–411 (1955).

The problem in question is the following: for which functions $\Phi(x)>0$ on $(-\infty,+\infty)$ is it true that the set $x^n/\Phi(x)$, $n=0,1,2,\cdots$, is fundamental relative to the uniform topology in the space C_0 of all functions continuous on $(-\infty,+\infty)$ and vanishing at infinity? The author [Proc. Amer. Math. Soc. **4**, 869–875 (1953); MR **15**, 407] and Ahiezer and Bernstein [Dokl. Akad. Nauk SSSR (N.S.) **92**, 1109–1112 (1953); MR **15**, 689] gave different necessary and sufficient conditions in order that Φ have the above property. Here the author gives a new proof that

$$(*)\qquad \sup\int_{-\infty}^{+\infty}\frac{\log^+|P(x)|}{1+x^2}dx=+\infty,$$

the supremum taken for all real polynomials $P(x)$ with $|P(x)|<\Phi(x)$ for $-\infty<x<+\infty$ (which is equivalent to the condition of the second cited paper), is necessary and sufficient. If C_0 is replaced by L^p on $(-\infty,+\infty)$, (*) is still necessary and sufficient, the limitation on $P(x)$ being now $\|P/\Phi\|<1$. Finally, $x^n/\Phi(x)$ are fundamental in the space of functions continuous on $(0,+\infty)$ and vanishing at $+\infty$ if and only if $x^n/\Phi(x^2)$ are fundamental in C_0. *G. G. Lorentz* (Detroit, Mich.).

Mergelyan, S. N. On weighted approximations by polynomials. Dokl. Akad. Nauk SSSR (N.S.) **97**, 597–600 (1954); erratum, **101**, 196 (1955). (Russian)

Let $h(x)$ be a bounded, non-negative function. Let C_h denote the Banach-space of functions $f(x)$ defined and continuous on the whole real axis and such that $h(x)f(x)\to 0$ as $|x|\to\infty$ with the norm $\|f\|=\sup|h(x)f(x)|$ $(-\infty<x<\infty)$. The following results are obtained, giving a new solution of the Bernstein approximation problem (for a solution in different terms see the paper reviewed above).

Let $M_h(z)=\sup|P(z)|$ where P runs through all polynomials with $\|P\|\leqq 1$ (C_h norm). (1) $S=\{x^n\}$ $(n=0,1,2,\cdots)$ is total in C_h, if and only if $M_{h(x)/(1+|x|)}(a)=\infty$ for some non-real number a. (2) If $M_h(a)=\infty$ for some non-real a, then $M_h(z)=\infty$ for all non-real z. (3) If $M_h(z)<\infty$ for non-real z, then $M_h(z)<\exp(|z|^{1+\epsilon})$ $(|z|>\text{Re}(\epsilon))$ and the closure of S in C_h consists of the entire functions satisfying $|f(z)|<M_h(z)\|f\|$. (4) Let E be a nowhere dense closed set dividing the z-plane into n unbounded domains $D_1,\cdots,D_n$. The space $C_h(E)$ is defined in obvious analogy to C_h. If there is an entire function $G(z)$ such that $|G(z)|<1$ $(z\in E)$, $\sup|G(z)|>1$ $(z\in D_k,\ k=1,2,\cdots)$, then the polynomials are total in $C_h(E)$, if, for every $p>0$, $m(2|z|)h(z)\to 0$ as $z\to\infty$ in E. Here $m(r)=\sup|G(z)|$ $(|z|=r)$.

Partial sketch of proof. By the subharmonicity of $\log M_h$, $M_h(a)=\infty$ implies $M_h(t)=\infty$ for a continuum T of t passing through a. If $t\in T$ and P is a polynomial with large $P(t)$, then $(P(t)-P(x))/P(t)(x-t)$ approximates $1/(x-t)$. Similarly for $1/(x-\bar{t})$. Since $\{1/(x-t),1/(x-\bar{t})\}$ is total in C_1 (i.e., $h\equiv 1$), it is total on C_h. This proves the sufficiency part of (1). The proof even shows totality in $C_{(1+|x|)h(x)}=C'$. If $Q(x)$ is an approximation polynomial of $1/(x-b)$ $(b\neq 0)$ in C', then $p(x)=[(x-b)Q-1]/\epsilon$ has bounded C_h-norm and

$|p(b)|$ is large. This proves (2) and the necessity of (1). The proof of (4) is similar to that of (1) with G^p (p large) replacing the polynomial P. The proof of (3) is based on an estimate of the form

$$\|1/(x-i\lambda)-q_\lambda(x)\|<b|a+ib|^{-3}+\phi(a,b)/M_k(a+ib)$$

$(1<|\lambda|<2,\ b>0,\ |a+ib|>3,\ q_\lambda$ a suitable polynomial, ϕ an explicitly computed function of a and b). ~~The necessity of (1) is not proved in the paper.~~ *W. H. J. Fuchs.*

Šaginyan, A. L. On approximation by polynomials. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki 7, no. 4, 1–21 (1954). (Russian. Armenian summary)

Etant donné un domaine Ω, Ω_p désigne l'espace de Banach des fonctions f analytiques dans Ω et telles que

$$\|f\|^p=\iint_\Omega |f(z)|^p d\sigma<\infty \quad (p>0).$$

L'auteur étudie le problème de la complétion (=densité) des polynômes dans Ω_p, lorsque Ω est un domaine non borné [pour $p=2$, voir S. N. Mergelyan, Uspehi Mat. Nauk (N.S.) 8, no. 4(56), 3–63 (1953), particulièrement les pages 43–53, où se trouvent exposés les résultats de l'auteur et de Džrbašyan; MR 15, 411]. Si Ω n'a pas de points communs avec un angle de mesure $\pi/\alpha<2\pi$ [c'est ainsi qu'il faut comprendre p. 2, l. 10 et p. 8, l. 18–19; note du référent] et coupe le cercle $|z|=r$ suivant un ensemble d'arcs de mesure linéaire $\exp(-\theta(r))$, il y a complétion lorsque $\int r^{-1-\alpha}\theta(r)\,dr=\infty$, avec quelques hypothèses de régularité sur $\theta(r)$; dans la démonstration, il est en outre supposé qu'une inversion transforme Ω en un "domaine de Carathéodory"; les hypothèses de régularité sont différentes de celles indiquées par Džrbašyan. Pour des domaines Ω plus particuliers, l'auteur donne un critère de normalité dans tout le plan de certaines familles de fonctions P_π appartenant à la boule unité de Ω_p, et il en tire un critère de non-complétion des polynômes dans Ω_p. En passant, il indique une condition nécessaire, portant sur la fonction poids, pour l'approximation polynomiale pondérée des fonctions continues sur une courbe non bornée. *J. P. Kahane* (Montpellier).

Šaginyan, A. L. On approximation in the mean by harmonic polynomials. Akad. Nauk Armyan. SSR. Dokl. 19, 97–103 (1954). (Russian. Armenian summary)

Le problème est ici celui de l'approximation en moyenne d'ordre $p>0$ des fonctions harmoniques et bornées dans le domaine Ω par des polynômes harmoniques. L'auteur montre d'abord que cette approximation est possible 1) si Ω est un domaine de Carathéodory, 2) si Ω est "du type lune" et que les polynômes sont denses dans Ω_p (pour les définitions, voir le mémoire de Mergelyan cité ci-dessus). Puis il applique ses propres résultats et ceux de Džrbašyan au cas où Ω est limité par un cercle C et une courbe tangente intérieurement à C. *J. P. Kahane* (Montpellier).

Shisha, Oved, Sternin, Chayim, and Fekete, Michael. On the accuracy of approximation to given functions by certain interpolatory polynomials of given degree. Riveon Lematematika 8, 59–64 (1954). (Hebrew. English summary)

Let $f(x)$ be real and bounded in $[-1, 1]$ and let $\mu(\delta)$ be its modulus of continuity there. Improving the approximation theorem of Weierstrass, D. Jackson [Dissertation, Göttingen, 1911) showed that there exist polynomials $P_n(x)$ of degree n ($n=1, 2, \cdots$) for which

$$\sup_{-1\leq x\leq 1}|f(x)-P_n(x)|=O(\mu(2/n))$$

as $n\to\infty$. The authors employ Jackson's method to obtain a similar improvement of a theorem of L. Fejér [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1916, 66–91], namely: Let $I_n(x)$ be the polynomial of least degree satisfying $I_n(x_j^{(n)})=f(x_j^{(n)})$, $I_n'(x_j^{(n)})=0$ for $x_j^{(n)}=\cos(2j-1)\pi/2n$ $(j=1, \cdots, n)$. Then $|f(x)-I_n(x)|\leq 5\mu(2/\sqrt{n})$ for all $-1\leq x\leq 1$. *A. Dvoretzky* (New York, N. Y.).

Greville, T. N. E., and Vaughan, Hubert. Polynomial interpolation in terms of symbolic operators. Soc. Actuar. Trans. 6, 413–476 (1954).

With the general interpolation formula

$$v_x=\sum_{n=-\infty}^{n=\infty} L(x-n)u_n$$

the authors associate the characteristic operator of the formula

$$G=\int_{-\infty}^{\infty} L(t)E^{-t}dt,$$

where E is the shift operator $Eu_n=u_{n+1}$. They give rules (justified in a mathematical appendix) by which to obtain the characteristic operator G of an interpolation formula having prescribed properties; the formula itself can then be obtained from G by inversion. Six useful tables are given to effect this process economically. The form of G is usually of surprising simplicity. For example, the ordinary central-difference interpolation formula to 5th differences has

$$G=M^6(1-\tfrac{1}{4}D^2+30D^4),$$

where D is the differentiation operator and the operator M is defined by

$$M=\int_{-1/2}^{1/2} E^t dt.$$

L. M. Milne-Thomson (Greenwich).

Special Functions

Henrici, Peter. Kleine Bemerkung zur asymptotischen Entwicklung des Fehlerintegrals. Z. Angew. Math. Phys. 6, 145–146 (1955).

The author writes down the (known) asymptotic expansion of

$$\int_z^\infty e^{-\frac{1}{2}t^2}dt, \quad |\arg z|\leq\pi,$$

for $z\to\infty$, and estimates the remainder term. He also corrects several statements in a paper by J. Zbornik [same Z. 5, 345–351 (1954); MR 16, 289]. *A. Erdélyi.*

Fempl, Stanimir. Généralisation d'une relation de Legendre. Srpska Akad. Nauka. Zb. Rad. 43, Mat. Inst. 4, 41–56 (1955). (Serbo-Croatian. French summary)

The author studies the expression

$$L(k,\phi)=F(k,\pi/2)E(k',\phi)+E(k,\pi/2)F(k',\phi)-F(k,\pi/2)F(k',\phi),$$

where F and E are the elliptic integrals of the first and second kinds, and k, k' are complementary moduli. He ex-

presses complete elliptic integrals of the third kind in terms of L, and gives transformation formulas for, and a geometrical interpretation of, $L(k, \phi)$. *A. Erdélyi.*

Emersleben, Otto. Über Summen Epsteinscher Zetafunktionen regelmässig verteilter "unterer" Parameter. Math. Nachr. 13, 59–72 (1955).

Proof of

$$Z\begin{vmatrix}0 & 0\\ x & y\end{vmatrix}(s)+Z\begin{vmatrix}0 & 0\\ \frac{1}{2}-x & \frac{1}{2}-y\end{vmatrix}(s)=2^{1-\frac{1}{2}s}Z\begin{vmatrix}0 & 0\\ x+y & x-y\end{vmatrix}(s)$$

and of

$$\sum_{m_1=1}^{n}\cdots\sum_{m_p=1}^{n}Z\begin{vmatrix}0 & \cdots & 0\\ \vdots & \cdots & \vdots\\ h_1+m_1/n & \cdots & h_p+m_p/n\end{vmatrix}(s)$$
$$=n^{p-s}Z\begin{vmatrix}0 & \cdots & 0\\ \vdots & \cdots & \vdots\\ nh_1 & \cdots & nh_p\end{vmatrix}(s)$$

with examples of such transformations. *A. Erdélyi.*

Meijer, C. S. Expansion theorems for the G-function. IX. Generating functions of generalized hypergeometric polynomials and functions. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17, 243–251 (1955).

[For parts I to VIII see MR 14, 469, 642, 748, 979; 15, 422, 791, 955.] The author lists three familiar generating functions of Laguerre polynomials and shows that the theorems developed in the earlier parts lead to vast generalizations of these generating expansions. Many known expansions turn up as special forms of the author's results.
A. Erdélyi (Pasadena, Calif.).

Meijer, C. S. Expansion theorems for the G-function. X. Generating functions of generalized hyper-geometric polynomials and functions. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17, 309–314 (1955).

Continuing the work of Part IX [reviewed above] the author derives an expansion of a generalized hypergeometric function in an infinite series of products of hypergeometric polynomials and hypergeometric functions, and shows that a number of known expansions are contained in his result.
A. Erdélyi (Pasadena, Calif.).

Srivastara, H. M. On the generalized K-function of Bateman. J. Math., Tokyo 1, 137–144 (1953).

See the review of an earlier paper by the same author [Bull. Calcutta Math. Soc. 44, 59–62 (1952); MR 15, 218].
A. Erdélyi (Pasadena, Calif.).

Chakrabarty, N. K., and Sarkar, G. K. On the generalised K-function of Bateman and an allied function. J. Math., Tokyo 1, 145–154 (1953).

The so-called generalized K function is a confluent hypergeometric function, and all results listed in this paper are paraphrases of formulas on confluent hypergeometric functions. The T-function is a particular solution of an inhomogeneous confluent hypergeometric equation with a constant right-hand side. *A. Erdélyi* (Pasadena, Calif.).

Koschmieder, Lothar. On certain determinants formed with Hermitian functions of the second kind. Tecnica. Rev. Fac. Ci. Ex. Tec. Univ. Nac. Tucuman. 1, 314–317 (1952). (Spanish. English summary)

Let $h_n=h_n(x)$ be the Hermite functions of the second kind [P. Appell and J. Kampé de Fériet, Fonctions hypergéométriques et hypersphériques, polynomes d'Hermite, Gauthier-Villars, Paris, 1926, pp. 357–362]. Let

$$\delta_n=\delta_n(x)=h_n^2-h_{n-1}h_{n+1}.$$

The author shows that

$$\delta_n=(n-1)!\delta_1+\sum_{k=1}^{n-1}(n-1)!h_k^2(k!)^{-1}.$$

Hence $\delta_1>0$ implies an inequality of Turán type $\delta_n>0$. The author shows that $\delta_1>0$ if and only if $|x|<\Xi\doteq 2.5$.
G. E. Forsythe (New York, N. Y.).

Slater, L. J. Integrals for asymptotic expansions of hypergeometric functions. Proc. Amer. Math. Soc. 6, 226–231 (1955).

The author uses contour integrals of a type introduced in an earlier paper [Proc. Cambridge Philos. Soc. 48, 578–582 (1952); MR 14, 372] to obtain asymptotic expansions of basic hypergeometric functions. In particular, she obtains an asymptotic expansion of ${}_1\Phi_1$ which is analogous to the well-known asymptotic expansion of the confluent hypergeometric function ${}_1F_1$. *A. Erdélyi* (Pasadena, Calif.).

Slater, L. J. The integration of hypergeometric functions. Proc. Cambridge Philos. Soc. 51, 288–296 (1955).

The author evaluates a Mellin-Barnes type integral the integrand of which is a product of gamma functions, a hypergeometric function, and z^s (s being the variable of integration). The results are expressed in terms of hypergeometric series in two variables; they extend some results of C. S. Meijer [Nederl. Akad. Wetensch., Proc. 49, 227–237, 344–356, 457–469, 632–641, 765–772, 936–943, 1063–1072, 1165–1175 (1946); MR 8, 156, 379; and other papers]. There are several examples illustrating the general formula, and a further theorem involving a product of two hypergeometric functions in the integrand. *A. Erdélyi.*

Slater, L. J. Some basic hypergeometric transforms. J. London Math. Soc. 30, 351–360 (1955).

The work described in the preceding review is carried over to basic hypergeometric series. *A. Erdélyi.*

***Petiau, Gérard. La théorie des fonctions de Bessel exposée en vue de ses applications à la physique mathématique.** Centre National de la Recherche Scientifique, Paris, 1955. 477 pp. 2500 francs.

In this book the author has collected much of the information available on Bessel functions, with particular regard to those properties of Bessel functions of interest to applied mathematicians. Much, but not all, of the recent literature is included in this monograph, and there are valuable references to numerical tables. The presentation ranges from detailed proofs of the principal properties to brief references to the literature in the case of less important items.

Contents: Bessel coefficients, Bessel functions of integral order. Bessel functions of order zero. Bessel functions of arbitrary order, series representations. Bessel functions of arbitrary order, integral representations. Asymptotic expansions of Bessel functions. Addition and multiplication theorems. Zeros of Bessel functions. Integrals involving Bessel functions. Neumann polynomials and series, Lommel functions, Kapteyn series. Functions related to Bessel functions. Bessel integral functions. Kelvin functions. Fourier-Bessel series and integrals. Schlömilch series. Orthogonal polynomials and Bessel functions. Differential equations whose solutions can be expressed in terms of Bessel func-

tions. Kepler's problem, Bessel functions of several variables. Bernoulli's problem. Applications of Bessel functions to random-walk problems. Heat-conduction problems. Vibration of a circular membrane, and of an elastic plate, finite Hankel transforms. Applications of Bessel functions to diffraction problems. Applications to electromagnetic wave propagation. Applications to wave mechanics. Numerical tables. Graphs. *A. Erdélyi* (Pasadena, Calif.).

Gatteschi, Luigi. Sugli zeri della derivata delle funzioni di Bessel di prima specie. Boll. Un. Mat. Ital. (3) **10**, 43–47 (1955).

Let $j'_{n,r}$ be the rth positive zero of $dJ_n(x)/dx$ $(0\leqq n\leqq 1)$, and set $x_{n,r}=(2r+n)\pi/2+\pi/4$. The author proves that

$$\left|j'_{n,r}-x_{n,r}+\frac{4n^2+3}{8x_{n,r}}\right|\leqq\frac{2.85}{(2r+n)^3}\quad(r=1,2,\cdots).$$

A. Erdélyi (Pasadena, Calif.).

Schottlaender, Stefan. Über die Transformation einer Reihe nach Besselschen Funktionen. Arch. Math. **6**, 275–280 (1955).

The author transforms

$$\sum_{n=0}^{\infty}e^{in\delta}J_\nu(y+na)$$

in an infinite series of elementary functions. ν is an integer, a, y, δ are complex parameters, $a\neq 0$, $0\leqq y/a\leqq 1$, $\operatorname{Im}\delta\geqq|\operatorname{Im}a|$. The special case $\nu=0$, $y=0$ was investigated by Hermann Schmidt (unpublished). *A. Erdélyi.*

Chako, Nicholas. On integral relations involving products of spheroidal functions. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-73**, i+36 pp. (1955).

In a general integral theorem for spheroidal functions [Meixner and Schäfke, Mathieusche Funktionen und Sphäroidfunktionen, Springer, Berlin, 1954, p. 32, Theorem 1; MR 16, 586], the author generates a kernel by applying to a product solution of the time-harmonic wave equation in spheroidal coordinates a differential operator which commutes with the Laplacian. Certain linear combinations of the linear and angular momentum operators, and powers and products of such combinations, are used in this context. With the kernels thus generated the author obtains a number of integral relations involving products of spheroidal functions, and as a limiting case, also integral relations involving Legendre functions. *A. Erdélyi.*

Krickeberg, Klaus. Über die asymptotische Darstellung der Aufspaltung von Paaren benachbarter Eigenwerte der Differentialgleichung der Sphäroidfunktionen. Z. Angew. Math. Phys. **6**, 235–239 (1955).

In this note the third term of the asymptotic expansion for the difference of a pair of adjacent eigenvalues of the differential equation for the spheroidal functions is given. *N. D. Kazarinoff* (Lafayette, Ind.).

Herz, Carl S. Bessel functions of matrix argument. Ann. of Math. (2) **61**, 474–523 (1955).

"In this work we generalize the classical special functions of hypergeometric type .... Particular stress is laid on a generalized Bessel function ... which is a complex-valued function having for argument a complex $m\times n$ symmetric matrix .... First, a large number of formulae from the classical theory of special functions are given appropriate generalizations. Some of these turn out to have applications to lattice-point problems and to the theory of non-central Wishart distributions in statistics. Secondly, the L^2-theory of the Hankel transform is established with the generalized Bessel functions furnishing the kernel .... The third class of results concerns the properties of harmonic polynomials in several variables .... They are related ... to generalized Gegenbauer polynomials ..." (From the author's Introduction.)

The principal tool in the author's investigations is the Laplace transform whose kernel is $\exp[-\text{trace }(\Lambda Z)]$, integration being extended over the set of all positive definite matrices Λ. If the integral converges in some "right half-plane", i.e., if $\operatorname{Re}Z-X_0$ is positive definite for some X_0, then the Laplace transform is an analytic function of the symmetric matrix Z. Starting with ${}_0F_0(\Lambda)=\exp(\text{trace }\Lambda)$, all generalized hypergeometric functions may be generated by a succession of Laplace transformations and inverse Laplace transformations. *A. Erdélyi.*

Eason, G., Noble, B., and Sneddon, I. N. On certain integrals of Lipschitz-Hankel type involving products of Bessel functions. Philos. Trans. Roy. Soc. London. Ser. A. **247**, 529–551 (1955).

In Part I the authors express

$$I(\mu,\nu;\lambda)=\int_0^\infty J_\mu(at)J_\nu(bt)e^{-ct}t^\lambda dt$$

as an integral of a hypergeometric function, simplify this integral in certain special cases, and give explicit values in terms of elliptic integrals for $(\mu,\nu;\lambda)=(0,0;0)$, $(1,1;0)$, $(1,1;\pm1)$, $(1,0;\pm1)$, and $(1,0;0)$. They also give expansions or approximate expressions for small b/a, or $b/a\approx 1$ and c/a small, or c small; and convert I into an integral involving modified Bessel functions. [Reviewer's remark: For related and partly overlapping results see G. N. Watson, Theory of Bessel functions, Cambridge, 1922, §§13.22, 13.23, 13.4; J. London Math. Soc. **9**, 16–22 (1934); W. N. Bailey, Proc. London Math. Soc. (2) **40**, 37–48 (1935).]

In Part II the authors describe the numerical computation of I and give 4D tables for $\mu=0,1$; $\nu=0,1$; $\lambda=0,\pm1$ (except for $\mu=\nu=0$, $\lambda=-1$ when the integral fails to exist). *A. Erdélyi* (Pasadena, Calif.).

Prache, Pierre M. La ligne à retard électromagnétique, source de relations entre les fonctions de Bessel et les intégrales elliptiques. Ann. Télécommun. **10**, 82–86 (1955).

Brusencov, N. P. On wave functions of the elliptic cylinder. Vestnik Moskov. Univ. **9**, no. 9, 23–31 (1954). (Russian)

The author investigates certain solutions of the associated Mathieu equation, Jc_n and Nc_n being solutions satisfying the differential equation corresponding to that of ce_n and reducing, respectively, to the Bessel function of the first kind J_n and that of the second kind N_n, when the eccentricity of the ellipse approaches zero; and similarly for Js_n and Ns_n. The paper contains integral relations between Jc_n and Js_n on the one hand and ce_n and se_n on the other hand, Bessel function expansions for Jc_n Js_n, Nc_n, Ns_n, and graphs comparing Jc_0, Nc_0, Jc_1, Js_1, Nc_1, Ns_1 with, respectively, J_0, N_0, J_1, N_1. *A. Erdélyi* (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Jackson, L. K. On generalized subharmonic functions. Pacific J. Math. 5, 215–228 (1955).

Beckenbach and Jackson [same J. 3, 291–313 (1953); MR 14, 1084] have generalized continuous subharmonic functions by replacing harmonic functions, as the dominating functions, by an abstract class $\{F\}$ of functions satisfying certain postulates. Here, this theory is extended to include upper semicontinuous sub-$\{F\}$ functions which are bounded on closed subsets of the domain of definition. Some of the principal theorems on subharmonic functions are generalized, and it is shown that the results of J. W. Green [Proc. Amer. Math. Soc. 3, 829–833 (1952); MR 14, 271] on approximately subharmonic functions can be extended to sub-$\{F\}$ functions. *F. F. Bonsall* (Newcastle-on-Tyne).

Arsove, Maynard G. The Looman-Menchoff theorem and some subharmonic function analogues. Proc. Amer. Math. Soc. 6, 94–105 (1955).

Remarques sur divers critères d'analyticité et sousharmonicité, groupées autour d'un procédé de démonstration simple, basé sur le théorème de Baire. Exemple: si u, continue dans un ouvert de R^n, a son laplacien généralisé inférieur $>-\infty$ sauf sur un F_σ de capacité nulle, et $\geqq 0$ presque partout, alors u est sousharmonique. Le résultat connu de Privaloff [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 31, 102–103 (1941); MR 2, 366] fait seulement intervenir le laplacien supérieur, mais l'ensemble exceptionnel est supposé fermé. Parallèlement, on donne une condition suffisante, voisine de celle de W. Rudin [Trans. Amer. Math. Soc. 68, 278–286 (1950); MR 11, 663], pour que les masses associées à une fonction sousharmonique soient absolument continues. *J. Deny* (Princeton, N. J.).

\# ***Brelot, Marcel. Topology of R. S. Martin and Green lines.** Lectures on functions of a complex variable, pp. 105–121. The University of Michigan Press, Ann Arbor, 1955. \$10.00.

R. S. Martin introduced [Trans. Amer. Math. Soc. 49, 137–172 (1941); MR 2, 292] an ideal boundary for open domains which is well suited for the investigation of bounded harmonic functions in the domain. In the present paper the author gives a detailed investigation of some of the properties of this ideal boundary. *H. L. Royden.*

\# ***Brelot, Marcel. Topologies on the boundary and harmonic measure.** Lectures on functions of a complex variable, pp. 85–103. The University of Michigan Press, Ann Arbor, 1955. \$10.00.

A central problem in the theory of harmonic functions in a domain in Euclidean space is that of topologizing the domain and its boundary in such a way that a certain class of harmonic functions are continuous in the closed domain and such that the Dirichlet problem has a solution for functions continuous on the boundary. The present paper is an exposition of work on this problem for the case of bounded harmonic functions. *H. L. Royden* (Stanford, Calif.).

order ***Brelot, Marcel, et Choquet, Gustave. Polynômes harmoniques et polyharmoniques.** Second colloque sur les équations aux dérivées partielles, Bruxelles, 1954, pp. 45–66. Georges Thone, Liège; Masson & Cie, Paris, 1955.

Let $\mathcal{P}_n$ be the vector space of all real homogeneous polynomials of degree n in ν variables. Consider the polynomials $P_n \in \mathcal{P}_n$ as defined on the Euclidean space R^ν, and norm $\mathcal{P}_n$ by setting $\|P_n\|^2=\int P_n^2 d\mu$, where μ is the rotation-invariant measure of total mass 1 carried by the unit sphere $S^{\nu-1}$ in R^ν. Let $\rho^2\mathcal{P}_{n-2}$ be the subspace of $\mathcal{P}_n$ which consists of all P_n divisible by $\rho^2=\sum_1^\nu x_i^2$. Then $\rho^2\mathcal{P}_{n-2}$ is invariant under all rotations of R^ν, and so is its orthogonal complement $\mathcal{H}_n$. Moreover, $\mathcal{P}_n$ is the direct sum of the spaces $\rho^{2p}\mathcal{H}_{2n-p}$ $(0\leqq 2p\leqq n)$, and these are the only minimal invariant subspaces of $\mathcal{P}_n$.

A $\mathcal{P}_n$-Laplacian is defined to be a linear mapping Δ of $\mathcal{P}_n$ into $\mathcal{P}_{n-2}$ which commutes with all rotations of R^ν. It is shown that $\Delta P=0$ for every $P \in \mathcal{H}_n$, and that the converse is true if Δ is the ordinary Laplacian. Thus H_n is identified as the set of all homogeneous harmonic polynomials of degree n; the basic properties of the harmonic polynomials are then obtained fairly easily.

There are several applications, of which the following two are samples. (1) If $A \in \mathcal{P}_2$ and if A has constant sign, then every polynomial P has a unique representation

$$P=H_0+AH_1+A^2H_2+\cdots,$$

where the polynomials H_p are harmonic. (2) If D is an ellipsoidal domain in R^ν and P is a polynomial on R^ν, then there is a harmonic polynomial H on R^ν which has the same degree as P and which coincides with P on the boundary of D; ellipsoidal domains are the only ones with this property. *W. Rudin* (Rochester, N. Y.).

\# ***Ullman, J. L. Regularity criteria in potential theory.** Lectures on functions of a complex variable, pp. 385–386. The University of Michigan Press, Ann Arbor, 1955. \$10.00.

Application du critère de Wiener au théorème suivant: le produit cartésien d'un ensemble plan de capacité logarithmique positive par le segment unité est de capacité newtonienne positive. Un résultat plus complet est établi par une autre méthode dans J. Deny et P. Lelong [Bull. Soc. Math. France 75, 89–112 (1947); MR 9, 352]. *J. Deny* (Princeton, N. J.).

du Plessis, N. Half-space analogues of the Fejér-Riesz theorem. J. London Math. Soc. 30, 296–301 (1955).

On considère l'espace euclidien à $n+1$ dimensions $(x_1, \cdots, x_n, z)$ et une fonction harmonique f dans le demi-espace $z>0$. On suppose

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}|f(x_1, \cdots, x_n, z|^r dx_1\cdots dx_n$$

bornée. Alors $f(x_1, \cdots, x_n)=\lim_{z\to 0} f(x_1, \cdots x_n, z)$ existe presque partout et

$$M_r(f)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}|f(x_1, \cdots, x_n)|^r dx_1\cdots dx_n<+\infty.$$

Puis

$$\int_0^{+\infty}|f(x_1, \cdots, x_n, z)|^r z^{n-1}dz,$$

et

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\int_0^{+\infty}|f(x_1, \cdots, x_n, z)|^r z^{m-2}dx_m\cdots dx_n dz$$

sont majorées par $M_r(f)$ à des constantes multiplicatives près (fonctions de r et n). On donne pour la première l'expression de la meilleure constante. *M. Brelot* (Paris).

Ozawa, Mitsuru. Some classes of positive solutions of $\Delta u = Pu$ on Riemann surfaces. II. Kōdai Math. Sem. Rep. 7, 15–20 (1955).

This paper gives an account of further results obtained by the author in the study initiated in part I [same Rep. 1954, 121–126; MR 16, 819]; §4 is a continuation of the study of the dimension of the ideal boundary; §5 treats dimension questions for subregions; in §6 another proof is given for the existence of a Green's function associated with the Poisson equation on a Riemann surface.
M. H. Heins (Providence, R. I.).

Babuška, Ivo. Bemerkung zur gewissen Lösung des biharmonischen Problems. Časopis Pěst. Mat. 79, 41–63 (1954). (Czech. Russian and German summaries)

If $u(x, y)$ satisfies in a domain Ω of the s-plane ($s=x+iy$) the biharmonic equation $\Delta^2u=0$ it can be written in the form $u=\mathrm{Re}\,\{\bar{z}\varphi(z)+\chi(z)\}$ with analytic functions $\varphi(z)$ and $\chi(z)$. Let ψ_n ($n=1, 2, \cdots$) be a sequence of entire functions whose real parts form a complete orthonormal system of harmonic functions in Ω, in the L_2-sense. If Δu is square integrable, it can be expanded in terms of the Re (ψ_n). The author discusses the computation of the coefficients of the expansion in terms of given boundary values of u and the normal derivative $\partial u/\partial v$ and the subsequent expressions for $\varphi(z)$ and $\chi(z)$. He shows that the formulas obtained by formal computation hold under rather general interpretation of the concept of boundary value and a wide assumption about the nature of the boundary of Ω. *C. Loewner.*

Orlov, A. A. On a method of expanding the force function of a compressed ellipsoid of revolution in a series of Legendre polynomials. Moskov. Gos. Univ. Trudy Gos. Astr. Inst. 24, 131–137 (1954). (Russian)

Verf. entwickelt das Potential eines abgeplatteten homogenen Rotationsellipsoids im Aussengebiet in eine Reihe nach Legendreschen Polynomen. Im Asnchluss hieran erhält er eine entsprechende Entwicklung für inhomogene Rotationsellipsoide, in deren Innerem die Dichte auf koaxial gelegenen Ellipsoiden konstant ist. *K. Maruhn.*

Kolbenheyer, T. Das geoelektrische Stromfeld in homogenen Halbraum in Anwesenheit eines kugelförmigen Fremdkörpers. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 4, 100–153 (1954). (Slovak. Russian and German summaries)

The author first obtains analytical expressions for the field of a (i) simple source, (ii) dipole, (iii) quadrupole and octapole, in an infinite medium which is homogeneous except for a sphere embedded in it. These fields are given by infinite series which may be interpreted as multipole expansions. After discussing the reflection of multipole waves on a plane surface, the author proceeds to obtain approximate expressions for waves in a half-space containing a spherical inclusion, it being assumed that the waves are generated by two sources situated on the boundary of the half-space while the centre of the sphere is in the plane of symmetry of the two sources. The theoretical results are applied to a discussion of geo-electrical prospecting. *A. Erdélyi.*

Kolbenheyer, Tibor. Der Einfluss einer halbkugelartigen Inhomogenität auf ein künstliches geoelektrisches Feld. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 4, 227–236 (1954). (Slovak. Russian and German summaries)

[See also the preceding review.] In this paper there is one source located on the boundary of a half-space, the half-space being homogeneous except for a hemispherical inclusion the centre of which is on the boundary of the half-space. *A. Erdélyi* (Pasadena, Calif.).

Differential Equations

Capriz, Gianfranco. Sulla determinazione delle linee integrali di un sistema differenziale che appartengono ad una assegnata ipersuperficie. Rend. Mat. e Appl. (5) 14, 533–541 (1955).

The author considers a system of equations
$$dx_i/dt = X_i(x|t) \quad (i=1, \cdots, n),$$
where the functions X_i are analytic in a domain C in the $(n+1)$-dimensional space of the points $(x_1, \cdots, x_n, t)$. A solution of the system is represented by a curve in C. Let $f(x|t)=0$ be an equation defining an analytic hypersurface Σ_0 in C. The problem considered here is that of determining the class Γ, possibly empty, of solutions which are represented by curves lying on Σ_0. This problem is solved by a method which depends upon the consideration of a sequence of systems of differential equations involving successively fewer dependent variables. At the end of the process of reduction a certain function of one variable, $\gamma_m(t)$, is determined. A necessary and sufficient condition that the class Γ be non-empty is that $\gamma_m(t)\equiv 0$. If this condition is satisfied, the class Γ consists of those solutions whose initial points $(x_1, \cdots, x_n, t)$ satisfy a certain set of equations $f(x|t)=0$, $\gamma_1(x|t)=0, \cdots, \gamma_{m-1}(x|t)=0$.

By way of an application of the results, the author discusses the motion of a rigid body, having a point 0, other than the centroid, fixed, and being subjected to a uniform gravitational field. The motions are found for which the instantaneous axis of rotation always lies in one of the principal planes of the inertial ellipsoid at 0. It is found that the class of such motions is composed of four subclasses which have been studied previously from other points of view. *L. A. MacColl* (New York, N. Y.).

Ważewski, T. Sur la structure de l'ensemble engendré par les intégrales non asymptotiques des équations différentielles. Bull. Acad. Polon. Sci. Cl. III. 3, 143–148 (1955).

The present paper starts a new line in the topological theory of the author concerning the asymptotic behavior of the solutions of a differential system (*) $du/dt=F(t, u)$, $u=(u_1, \cdots, u_n)$, of n differential equations [see T. Ważewski, Ann. Soc. Polon. Math. 20, 279–313 (1948); MR 10, 122; F. Albrecht, Bull. Acad. Polon. Sci. Cl. III. 2, 315–318 (1954); MR 16, 248; A. Pliś, ibid. 2, 415–418 (1955); MR 16, 700; K. Tatarkiewicz, Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 7, 19–81 (1954); MR 16, 821]. According to his own usual notations let $\omega\subset\Omega$ be open sets of points $P=(t, u)$ in E_{n+1}, B the boundary of ω in Ω, S, E, G the subsets of B of "strict egress" (ends), "strict ingress", "external contact" for the trajectories of (*) in ω. Suppose that F is continuous in Ω, that a uniqueness theorem holds in Ω, that $B=S+E+G$, $S\neq 0$, $G\neq 0$, G closed, and that each point $P\in\omega+B$ is a point of a trajectory entering ω at some point of E. If Z is the set covered by all the trajectories entering ω at E and remaining there indefinitely (asymptotic integrals), let $\Gamma=\omega-Z$. Suppose that $du_i/dt_i=F_i(t_i, u_i)$, $i=1, 2$, are any two systems as above and that $\omega_i, \Omega_i, B_i, S_i, E_i, G_i, Z_i, \Gamma_i$ are the corresponding sets. Then the following theorem holds: If there exists a homeomorphism h

between S_1+G_1 and S_2+G_2 with $S_2=h(S_1)$, $G_2=h(G_1)$, then there is also a homeomorphism m between Γ_1 and Γ_2 with $S_2=m(S_1)$, $G_2=m(G_1)$, $E_2\Gamma_2=m(E_1\Gamma_1)$, $\omega_2\Gamma_2=m(\omega_1\Gamma_1)$, $\Gamma_2=m(\Gamma_1)$. The present theorem and others supplant previous ones where the concept of retraction was used. Applications are announced concerning the dimension of the set Z. *L. Cesari* (Lafayette, Ind.).

Lyaščenko, N. Ya. On a separation theorem for a linear system of differential equations with almost periodic coefficients. Ukrain. Mat. Ž. 7, 47–55 (1955). (Russian)

The author considers a system of linear differential equations of the form

$$dz_1/dt=A_1(t)z_1+L_{11}(t)z_1+L_{12}(t)z_2,$$
$$dz_2/dt=A_2(t)z_2+L_{21}(t)z_1+L_{22}(t)z_2,$$

where z_1 and z_2 are vectors and the coefficient matrices are almost periodic. Under certain assumptions concerning these matrices, it is shown that a change of variable converts this single system into two separate systems, each of lower order than the original. *R. Bellman.*

Petrovskiĭ, I. G., and Landis, E. M. On the number of limit cycles of the equation $dy/dx=M(x,y)/N(x,y)$, where the M and N are polynomials of second degree. Dokl. Akad. Nauk SSSR (N.S.) 102, 29–32 (1955). (Russian)

Outline of proof that the real equation

$$\text{(1)}\qquad dy/dx=M(x,y)/N(x,y),$$

where M, N are real quadratic polynomials, has at most three limit-cycles. As Bautin has shown [Mat. Sb. N.S. 30(72), 181–196 (1952); MR 13, 652] that there are systems (1) with three limit-cycles, three is the true upper bound of the number of limit-cycles.

The method of the authors rests upon the following most interesting considerations. Let R_4 be the complex projective completion of the space of the complex variables x, y (the infinite region is a complex line = a sphere). Let Φ denote a complete complex solution $y=\varphi(x)$ of (1) in R_4. One augments Φ by the poles and the branch-points of finite order of φ. An oriented Jordan curve on Φ is called a cycle. Lemma 1. A real oriented limit-cycle of (1) is a cycle not ~ 0 on the appropriate Φ. Two limit-cycles on the same Φ are not homologous to one another.

A cycle on Φ is said to be simple if its projections on the (complex) planes x, y are Jordan curves. Cycles $L_1, \cdots, L_k$ on various Φ's are said to be properly situated whenever no two of their projections on the x-plane intersect. Lemma 2. The number of limit-cycles of (1) does not exceed the maximum number of properly situated simple cycles that are neither ~ 0 nor to one another.

Using now continuity and limiting relations in the space of the coefficients of P, Q, the asserted result follows. *S. Lefschetz* (Princeton, N. J.).

Moon, Parry, and Spencer, Domina Eberle. On the specification of Bôcher equations. J. Franklin Inst. 260, 41–46 (1955).

The Bôcher equations

$$\frac{d^2Z}{dz^2}+P(z)\frac{dZ}{dz}+Q(z)Z=0 \quad (P,\, Q \text{ rational functions of } z)$$

provide a convenient foundation for building a classification of the ordinary differential equations of mathematical physics, showing when two apparently distinct equations are actually equivalent and have the same form of solution. The authors introduce the restrictions that

$$P(z)=\frac{1}{2}\left[\frac{m_1}{z-a_1}+\frac{m_2}{z-a_2}+\cdots+\frac{m_{n-1}}{z-a_{n-1}}\right],$$

$$Q(z)=\frac{A_0+A_1z+A_2z^2+\cdots+A_lz^l}{(z-a_1)^{m_1}(z-a_2)^{m_2}\cdots(z-a_{n-1})^{m_{n-1}}},$$

where n = number of distinct singular points (one at infinity); m_i, l = integers. The specification of those Bôcher equations is a sequence $\{m_1, m_2, \cdots, m_{n-1}; m_n\}$ of integers representing the orders of the poles, where m_n is the order of the pole at ∞. This method of specification leads to a convenient way of classifying the ordinary differential equations of field theory, e.g.

n				
4	cyclical equation II	{1222}		
	cyclical equation I	{1122},	{1121}	
	Lamé equation	{1113},	{1112},	{1111}
3	Legendre wave equation	{123},	{ 122},	{121}
	Paraboloidal wave equation	{114},	{113},	
2	Bessel wave equation	{24},	{23},	{22}
	Weber equation	{14},		
	Elementary	{33},	{32}	
1	Elementary	{04},	{01}	

A second table contains cases, where transformation of a Bôcher equation may yield a Bôcher equation. Finally the authors discuss how the new specification differs from that of Ince [Ordinary differential equations, Longmans-Green, London, 1927, Chap. XX]. *M. Pinl* (Cologne).

Sugiyama, Shohei. On the singularities of the differential equation

$$\frac{d^2y}{dx^2}+f(x,y)\frac{dy}{dx}+g(x,y)=P(x).$$

Kōdai Math. Sem. Rep. 7, 23–29 (1955).

In the equation

$$\text{(1)}\qquad \frac{d^2y}{dx^2}+f(y)\frac{dy}{dx}+g(y)=P(x),$$

$f(y)$ and $g(y)$ are polynomials of degree n and m respectively and $P(x)$ is a regular and single-valued function in a certain domain D. Suppose further that we can continue analytically a solution of (1) up to a finite point x_0 along any curve from a point at which the solution is regular but not beyond x_0. It is proved that, if the solution tends to ∞ as we approach to x_0, there exists a solution of (1) of the form:

$$\text{i)}\qquad y=\sum_{\nu=-1}^{\infty}a_\nu(x-x_0)^{\nu/n}$$

if $n\geq m-1$;

$$\text{ii)}\qquad y=\sum_{\nu=-1}^{\infty}a_\nu(x-x_0)^{2\nu/(m-n-1)}$$

if $n<m-1$. Further, in order that x_0 not be a branch point, but a pole, it is necessary and sufficient that: i) $n\geq 2$, $m=n+2$; ii) $n=1$, $m=0$, 1, 2, 3, 4; iii) $n=0$, $n=2$, 3. Similar results are proved for the equations of the form

$$\frac{d^2y}{dx^2}+\frac{1}{x}f(y)\frac{dy}{dx}+g(y)=P(x)$$

and

$$\frac{d^2y}{dx^2}+\frac{1}{x}f(y)\frac{dy}{dx}+\frac{1}{x^2}g(y)=P(x).$$

M. Zlámal (Brno).

Volpato, Mario. Sull'esistenza e unicità di soluzioni periodiche per equazioni differenziali ordinarie del secondo ordine. Ann. Univ. Ferrara. Sez. VII. (N.S.) 3, 99–111 (1954).

The author first proves two theorems concerning the existence and uniqueness of a solution of the problem

$$\ddot{x}+k^2x=F(t, x, \dot{x}, \ddot{x}),\quad x(0)=x(T),\quad \dot{x}(0)=\dot{x}(T),$$

where all of the constants and variables are real, and the function F is continuous with respect to its four arguments. He then assumes that F is periodic with respect to t, with period T, and that kT is not an integral multiple of 2π, and proves two theorems concerning the existence and uniqueness of solutions of the differential equation which are periodic with the period T. All of the theorems involve numerous further conditions, and are too complicated to state explicitly here. However, the only condition which is severely restrictive, and which would not be expected as a matter of course, is one stating that to each positive number ρ there corresponds a positive number M_ρ such that in the domain

$$0\leqq t\leqq T,\quad |x|\leqq \rho|k|^{-1},\quad |\dot{x}|\leqq \rho,\quad |\ddot{x}|\leqq \rho|k|+M_\rho$$

we have the relation $|F|\leqq M_\rho$, and such that for large values of ρ we have the relation

$$(|\sin \tfrac{1}{2}kT|^{-1}+1)M_\rho T\leqq \rho.$$

In the case in which F does not involve $\ddot{x}$, the theorems admit of modified statements which are given in full.

L. A. MacColl (New York, N. Y.).

Wintner, Aurel. On linear instability. Quart. Appl. Math. 13, 192–195 (1955).

Let $f^+(t)$ and $F^+(t)$ denote max $(0, f(t))$ and $\int_{t_0}^{t} f^+(s)ds$ respectively. It is proved that the condition

$$\liminf_{t\to\infty} t^{-1}F^+(t)=0$$

is sufficient in order that the equation $x''+f(t)x=0$ have some solution satisfying $\limsup_{t\to\infty} |x(t)|=\infty$.

M. Zlámal (Brno).

Capra, Vincenzo. Sulle vibrazioni libere di un sistema meccanico ad un grado di libertà. Univ. e Politec. Torino. Rend. Sem. Mat. 13, 307–325 (1954).

Let $q(t)$ be a periodic solution of period T of $\ddot{q}+f(q)=0$, $q(0)=q_0$, $\dot{q}(0)=\dot{q}_0$. $f(q)$ is assumed to be piecewise continuous. If $\bar{f}(q)$ has the same zeros and discontinuities of $f(q)$ and if $|f(q)-\bar{f}(q)|<\epsilon f(q)$, then $\ddot{\bar{q}}+\bar{f}(\bar{q})=0$, $\bar{q}(0)=q_0$, $\dot{\bar{q}}(0)=\dot{q}_0$ has a periodic solution of period $\bar{T}$ such $\bar{T}=T+O(\epsilon)$ and $\bar{q}(t)=q(t)+O(\epsilon)$.

C. R. De Prima.

Tricomi, Francesco G. Equazioni differenziali con punti di transizione ("turning points"). Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 17, 137–141 (1954).

The author studies the asymptotic behavior of solutions of the differential equation

$$(1)\qquad \frac{d^2z}{dx^2}+[\mu^2x-Q(x)]z=0$$

for large values of μ. He re-writes (1) in the form

$$\frac{d^2z}{dt^2}+\tfrac{1}{3}tz=3^{2/3}\mu^{-4/3}Q(3^{-1/3}\mu^{-2/3}t)z$$

and then applies the method of Fubini [Tricomi, Equazioni differenziali, 2d ed., Einaudi, Torino, 1953, p. 189; MR 15, 793] to obtain the first two terms in the asymptotic expansion of z. He also shows how more general equations with transition points can be reduced to the form (1).

A. Erdélyi (Pasadena, Calif.).

Feščenko, S. F. Asymptotic solution of an infinite system of differential equations with slowly varying parameters. Dopovidi Akad. Nauk Ukrain. RSR 1954, 82–86 (1954). (Ukrainian. Russian summary)

The infinite system

$$(1)\qquad \frac{d^2z_n}{dt^2}+\omega_n^2z_n=\epsilon\sum_{j=1}^{\infty}A_{nj}(\tau)z_j+\epsilon B_n(\tau)e^{i\theta}$$

is considered, where ϵ is a small parameter and the ω_n are real numbers such that $\omega_n\to\infty$ as $n\to\infty$. The complex functions $A_{nj}(\tau)$ and $B_n(\tau)$ are assumed to possess derivatives with respect to τ of sufficiently high order in an interval $0\leqq\tau\leqq L$, and are such that

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\left|\frac{d^\nu A_{nj}(\tau)}{d\tau^\nu}\right|^2<\infty,\quad \sum_{n=1}^{\infty}\frac{1}{\omega_n^2}\left|\frac{d^\nu B_n(\tau)}{d\tau^\nu}\right|^2<\infty$$

for $\nu=0, 1, 2, 3, \cdots$. Solutions of the system (1) are found for the resonance and non-resonance cases. The resonance case is defined to be the case when the function $d\theta/dt=k(\tau)$ becomes equal to one of the numbers ω_n for some value of τ in $0\leqq\tau\leqq L$. The non-resonance case occurs when the function $d\theta/dt=k(\tau)$ is never equal to any one of the numbers ω_n.

H. P. Thielman (Ames, Iowa).

Sakurai, Akira. On extraordinary phenomena in a nonlinear forced oscillation. J. Phys. Soc. Japan 10, 274–278 (1955).

The equation $\ddot{x}-dF(x)/dx=p\cos(\omega t)$ is analysed by replacing it by a variation principle and inserting a Fourier expansion of period $2\pi/\omega$ for x. Equations for the coefficients are determined by a Rayleigh-Ritz technique. Approximations are made, but not justified, to simplify this system. An infinite determinant is constructed to whose roots correspond resonant frequencies. It is noted that the Fourier coefficients are generally real, but may become complex with a change of phase at resonance. It is remarked that the solution may become discontinuous as resonance is passed through, but details are not given. The theory is applied to a simple electrical circuit containing an iron-core inductor.

E. Pinney (Berkeley, Calif.).

Mitropol'skiĭ, Yu. A. On passage through a resonance of second order. Ukrain. Mat. Ž. 7, 121–123 (1955). (Russian)

The basic equation with forcing term is first put in the form

$$(1)\qquad \ddot{x}+x=\epsilon f(x, \dot{x})+E\sin nt.$$

Set $\xi=(\nu^2-n^2)/n^2$. In order to study the passage through n-uple resonance the author replaces (1) by

$$\ddot{x}+x=\epsilon(\tau)f(x, \dot{x}, \xi(\tau))+E\sin nt,\quad \tau=\epsilon t.$$

Essentially however ϵ varies little and may be assumed constant. In (2) he makes the change of variable

$$x=z+E(1-n^2)^{-1}\sin nt$$

which yields

$$(3)\qquad \ddot{z}+z=\epsilon\tau f(z+E(1-n^2)^{-1}\sin nt,\ \dot{z}+nE(1-n^2)^{-1}\cos nt, \xi(\tau)),$$

to which he may now apply the approximation scheme of an earlier paper [Prikl. Mat. Meh. 14, 139–170 (1950); MR 12, 181].

S. Lefschetz (Princeton, N. J.).

Barbălat, I. Une propriété globale des trajectoires d'un système d'équations différentielles équivalent à l'équation des oscillations non linéaires de Liénard. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 853–860 (1954). (Romanian. Russian and French summaries)

This paper extends a result of the reviewer on the equation of van der Pol [Contributions to the theory of non-linear oscillations, v. 2, Princeton, 1952, pp. 61–73; MR **14**, 557] as follows. Given the Liénard equation

$$(1)\qquad \ddot{x}+f(x)\dot{x}+x=0$$

or equivalently the system

$$(2)\qquad \begin{cases}\dot{x}=y-F(x),\\ \dot{y}=-x,\end{cases}\qquad F=\int_0^x f(\xi)\,d\xi,$$

suppose that: (a) f is continuous for every x; (b) there is an $a>0$ such that $xF(x)>0$ outside $|x|\geqq a$; likewise $f(x)\geqq 0$ for $|x|\geqq a$; (c) arbitrarily near $x=0$ there are values for which $F(x)\neq 0$; (d) for some $b>a$, $F(x)/x>1$ when $x>b$; (e) $F(x)/x$ is increasing for $x>0$, but decreasing for $x<0$. Under these circumstances there is a unique limit-cycle γ and every path of (2) (origin excepted) $\to\gamma$ as $t\to\infty$.

In his proof the reviewer leaned heavily upon the "polynomial" properties of his system. A different approach has been required here. [Additional reference: G. E. H. Reuter, Proc. Cambridge Philos. Soc. **47**, 49–54 (1951); MR **12**, 827.] *S. Lefschetz* (Princeton, N. J.).

Minorsky, Nicolas. Sur la résonance non linéaire. C. R. Acad. Sci. Paris **240**, 2482–2484 (1955).

The author employs the stroboscopic method to discuss the solutions of the equation $\ddot{x}-(a-cx^2)\dot{x}+x=\gamma\sin\omega t$, where a, c, γ are small parameters, and the value of the parameter ω is close to 1. He interprets the results as furnishing a tentative explanation of the fact that resonance curves for nonlinear systems are often found to have an approximately rectangular form. *L. A. MacColl.*

Cohen, Hirsh. On subharmonic synchronization of nearly-linear systems. Quart. Appl. Math. **13**, 102–105 (1955).

A perturbation method for calculating solutions of period $2\pi/\omega$ of the equation $\ddot{y}+\epsilon f(y)\dot{y}+\omega_0^2y=A\cos n\omega t$, where ϵ and $\omega-\omega_0$ are small, is presented. The special case $f(y)=y^2-1$, $n=3$, was previously treated by the author [Publ. Sci. Tech. Ministère de l'Air, Paris no. **281**, 169–187 (1953); MR **15**, 313]. *G. E. H. Reuter* (Manchester).

Reissig, Rolf. Über die Differentialgleichung

$$\frac{d^2x}{d\tau^2}+2D\cdot\frac{dx}{d\tau}+\mu\cdot\mathrm{sgn}\,\frac{dx}{d\tau}+x=\Phi(\eta\tau),$$

wo $\Phi(\eta\tau+2\pi)\equiv\Phi(\eta\tau)$ ist. Das Verhalten der Lösungen für $\tau\to\infty$. Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Nat. **1953**, no. 1, 33 pp. (1954).

The differential equation being studied is related to a mechanical system of one degree of freedom with viscous friction ($D>0$), Coulomb damping ($\mu>0$), and an elastic restoring force. The absence of a solution to the differential equation through a point of the phase space in the plane $\dot{x}=0$ corresponds to the mechanical system coming to a stop. The system may remain at rest for a while and then start up again. The principal result is: if the differential equation has a solution that for all time (τ) proceeds without stopping, then the system has a unique periodic, non-stopping solution; the period of the periodic solution is that of the forcing function Φ and all other non-stopping solutions approach the periodic solution as $\tau\to+\infty$. *J. P. LaSalle* (Notre Dame, Ind.).

Reissig, Rolf. Erzwungene Schwingungen mit zäher und trockener Reibung. Math. Nachr. **11**, 345–384 (1954).

This paper extends the results of the paper reviewed above to solutions with stops. It is shown that if the sliding friction is not too large ($2\mu<\max\Phi-\min\Phi$) all trajectories in the phase space approach a unique trajectory (the steady-state solution) as $\tau\to+\infty$. If the periodic solution is without stops, its period is that of the forcing function. If the periodic solution includes stops, the possibility appears to exist that the period is a fraction of that of the forcing function. *J. P. LaSalle* (Notre Dame, Ind.).

Reissig, Rolf. Erzwungene Schwingungen mit zäher und trockener Reibung (Ergänzung). Math. Nachr. **12**, 249–252 (1954).

In this supplement to the paper reviewed above a simplification in the proof of the principal result is achieved by the use of general properties of "stable" solutions due to E. Trefftz [Math. Ann. **95**, 307–312 (1925)]. *J. P. LaSalle* (Notre Dame, Ind.).

Reissig, Rolf. Erzwungene Schwingungen mit zäher und trockener Reibung. Abschätzung der Amplituden. Math. Nachr. **12**, 283–300 (1954).

Upper and lower bounds are given for the steady-state solution to the differential equation studied in the papers reviewed second and third above. *J. P. LaSalle.*

Reissig, Rolf. Erzwungene Schwingungen mit zäher Dämpfung und starker Gleitreibung. II. Math. Nachr. **12**, 119–128 (1954).

The differential equation

$$(1)\qquad \ddot{x}+g(\dot{x})+\mu\,\mathrm{sgn}\,\dot{x}+f(x)=\Phi(\omega t)$$

is considered subject to the following restrictions: (a) Φ is a continuous, periodic function of period 2π; (b) f is a continuous, monotone strictly-increasing function; $xf(x)>0$ for $x\neq 0$; and $2f(a)=\max\Phi+\min\Phi$ for some a; (c) g is a continuous, monotone increasing function and $xg(x)>0$ for $x\neq 0$; (d) $2\mu\geqq\max\Phi-\min\Phi$. If $u(t)$ is a solution to (1) so defined when $\dot{u}(t)=0$ that it describes physical behaviour, it is shown that $\dot{u}(t)\to 0$ and $u(t)\to u_0$ as $t\to+\infty$ (every solution approaches a rest position if the dry friction is sufficiently large). *J. P. LaSalle* (Notre Dame, Ind.).

Fragner, Wolfram. Über einen neuen Typ von Differentialgleichung und eine neue Methode zur Integration der linearen, quadratischen und kubischen Differentialgleichung. J. Reine Angew. Math. **194**, 180–182 (1955).

Under a simple transformation of the dependent variable, the differential equation $y'=f(x)+g(x)e^{-ay}$ goes over into an equation which can be solved by elementary methods. The author sets forth the general procedure, and discusses various applications and special cases. *L. A. MacColl.*

✶Simonart, Fernand. Sur une classe d'équations linéaires et de Riccati. IIIe Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 18–19. Fédération belge des Sociétés Scientifiques, Bruxelles.

The author determines all linear differential equations of the first order, and all Riccati equations, which define iso-

thermal families of curves. It is found that in all cases the solutions of these equations can be expressed in finite form in terms of elementary functions. *L. A. MacColl.*

Mitrinovitch, D. S. Sur une équation différentielle du premier ordre. Jber. Deutsch. Math. Verein. **58**, Abt. 2, 1 (1955).

It is shown how $(y')^s+\alpha xy^{n-1}y'+\beta y^n=0$ may be integrated by elementary means.

Aseltine, J. A. A transform method for linear time-varying systems. J. Appl. Phys. **25**, 761–764 (1954).

It is proposed to treat second-order differential equations by means of appropriate transforms. One example employs a type of Hankel transform studied by Meijer.

R. J. Duffin (Pittsburgh, Pa.).

Brillouet, Georges. Représentation de certaines intégrales d'équations différentielles. C. R. Acad. Sci. Paris **240**, 2113–2115 (1955).

The author gives a contour integral representation of particular solution of $Lu(z)=g(z)$, where $L\equiv\prod_{i=1}^{n}(d/dz-\rho_i)$, ρ_i are constants, and $g(z)=z^{-\lambda}$. Application is made to a problem of surface waves. *C. R. DePrima.*

Czajkowski, J., and Tietz, T. A note on the hypergeometrical differential equation. Prace Mat. **1**, 162–164 (1955). (Polish. Russian and English summaries)

Putnam, C. R. Integrable potentials and half-line spectra. Proc. Amer. Math. Soc. **6**, 243–246 (1955).

The differential equation $x''+(\lambda-f)x=0$ with f real and continuous on $0\leqq t<\infty$ is shown to be in the limit-point case if $\lim\int_0^T f(t)dt$ exists as $T\to\infty$. Moreover, the set S' of cluster points of the spectrum associated with

$$x(0)\cos\alpha+x'(0)\sin\alpha=0$$

consists of $0\leqq\lambda<\infty$. *N. Levinson* (Cambridge, Mass.).

Kay, Irvin. The inverse scattering problem. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-74**, i+29 pp. (1955).

The differential equation $u''+[k^2-V(x)]u=0$ is considered for $-\infty<x<\infty$. The solution has the asymptotic form $e^{ikx}+b(k)e^{ikx}$ as $x\to-\infty$ and $a(k)e^{ikx}$ as $x\to\infty$ and it is assumed the reflection coefficient $b(k)$ is known. From $b(k)$ the function $V(x)$ is determined (assuming $V=0$ for x less than some constant) by the method of Gelfand and Levitan for the case of the differential equation on a half-line. The differential equation $u''+k^2n^2(x)u=0$ is also considered on $-\infty<x<\infty$. It is assumed $n(x)$ is a constant for x less than a given value and $n(x)$ is determined from a knowledge of the reflection coefficient. *N. Levinson.*

Partial Differential Equations

Rutman, M. A. Spectral criteria of stability according to Lyapunov for some systems of linear partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **101**, 993–996 (1955). (Russian)

The author's previous results [MR **16**, 1126] on operator equations on a partially ordered space are applied to the stability theory of linear differential and partial differential equations in Banach space. Let $\bar{E}$ be the set of continuous functions $x(t)$, $0<t<\infty$, taking values in a complex Banach space $\bar{\mathcal{E}}$ and write $x>0$ if, for each t, $x(t)$ can be written in form $x(t)=\lambda(\xi+w)$, where λ is a positive scalar, ξ is a fixed element of unit norm, w any element of norm not exceeding a fixed $q<1$. Theorem 1 is typical: Let A be a bounded linear operator on $\bar{\mathcal{E}}$. In order that the solution of $dy/dt-Ay=x_1(t)$; $y(0)=x_0$ $(x_1\in\bar{E},\ x_0\in\bar{\mathcal{E}})$ should be bounded for arbitrary $x_0\in\bar{\mathcal{E}}$ and bounded $x_1\in\bar{E}$ it is necessary and sufficient that the spectrum of A lie in the left half-plane.

J. G. Wendel (Ann Arbor, Mich.).

Cinquini, Silvio. Un teorema di unicità per sistemi di equazioni a derivate parziali del primo ordine. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17**, 188–191 (1954); **17** (1954), 339–344 (1955).

In an earlier work [Ann. Mat. Pura Appl. (4) **32**,121–155 (1951); MR **13**, 845] existence and uniqueness theorems were established for the equation $z_x=f(x, y, z, z_y)$. Under an appropriate Lipschitz condition, a uniqueness theorem is presented (p. 189) for the system $z_{ix}=f_i(x, y, z_j, z_{ky})$ $(i, j, k=1, \cdots, r)$ with data $z_j(0, y)=g_j(y)$ in the special case in which $k=i$. The basic lemma (p. 339) is proved by arguments similar to those of the earlier work.

F. A. Ficken (Knoxville, Tenn.).

★**Schlegelmilch, Werner. Die Differentialoperationen der Vektoranalysis und ihre Bedeutung in Physik und Technik.** Verlag Technik, Berlin, 1954. xi+255 pp.

This book purports to bridge the gap between the treatment of vector analysis as a theory of pure mathematics and a tool of applied mathematics. Part I deals mainly with vector algebra and introductory vector analysis, with some applications to geometry and kinematics.

Part II contains a discussion of the first-order differential operators: the gradient, the divergence and the curl. These are introduced in connection with physical applications such as velocity, electrostatic and magnetic fields. Many types of physical problems are discussed, including material in hydrodynamics (Euler's and Bernoulli's equations), electromagnetic theory (Maxwell's equations and other topics) and the elements of potential theory. This part also contains a discussion of curvilinear coordinates, including a long list of data (in most cases without detailed derivation) for various coordinate systems.

Part III is devoted to the differential operators of higher order, such as the Laplacian. Expansions in Fourier series and in series of Legendre polynomials are introduced formally. Spherical harmonics are treated. There is a formal treatment of the first boundary-value problem of potential theory, using Green's function. This is specialized to obtain Poisson's integral for the circle and sphere. Green's function for the region between two concentric spheres is also studied. The chapter concludes with a brief mention of the use of inversion in potential theory. A short bibliography is appended.

The book gives an intuitive appreciation of the physical significance of some aspects of vector analysis, but throughout much of the discussion the treatment is formal, with little attention to the requirements of mathematical rigor. Infinitesimals are used, sometimes without adequate consideration of the limiting processes involved, and relevant hypotheses are not always given explicitly. The discussions leading up to the formulation of Gauss' divergence theorem and Stokes' theorem (pp. 69–77) illustrate these deficiencies.

The distinction between necessity and sufficiency of conditions is sometimes overlooked, as in the discussion of the

connection between div $\mathfrak{v}=0$ and $v=\text{rot}\,\mathfrak{B}$ on page 85. There are some typographical errors, not all of which are given in the list of errata. *F. W. Perkins.*

★Bers, L. Function-theoretical properties of solutions of partial differential equations of elliptic type. Contributions to the theory of partial differential equations, pp. 69–94. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

This paper discusses function-theoretic properties of solutions of elliptic equations of the form (*) $\Delta u+\alpha u_x+\beta u_y=0$ and of elliptic systems of the form (**) $w_{\bar{z}}=aw+b\bar{w}$. The author shows how the theory of pseudo-analytic functions may be applied to the study of second-order elliptic equations in two variables. The theory has been developed in the author's text, Theory of pseudo-analytic functions [Inst. Math. Mech., New York Univ., 1953; MR 15, 211]. Instead of attempting a complete description of the theory, the author confines himself to the proof of several fundamental facts and the statement of many theorems, complete proofs of which are given elsewhere. However a new proof is given of the basic theorem of Carleman on the unique continuation property of solutions of second-order elliptic equations in two variables. A complete proof is given of the similarity principle which, briefly stated, asserts that to every C' solution w of (**) (pseudo-analytic function) in a domain D there corresponds an analytic function f and a function $s(z)$, continuous in $\bar{D}$, such that $w(z)=e^{s(z)}f(z)$. (†) Conversely, to each analytic f there corresponds a w, solution of (**) with relation (†).

With the aid of the similarity principle several classical problems for (*) are shown to follow readily. In this manner a fundamental solution of (*) in the whole plane is obtained; existence of a Green's function and existence and continuity of the normal derivative of the Green's function are found at once. From this the solution of the Dirichlet problem is an immediate consequence.

From the similarity principle it is shown, by using functions similar to the powers $\{z^n\}$, that Taylor and Laurent expansions for pseudo-analytic functions in terms of "formal global powers" can be obtained. Isolated singularities are classified in the same manner as for analytic functions.

The final sections discuss quasi-linear equations of the form

$$(***)\qquad A(p,q)\varphi_{xx}+2B(p,q)\varphi_{xy}+C(p,q)\varphi_{yy}=0,$$

where $p=\varphi_x$, $q=\varphi_y$. Under a suitable ellipticity condition and mild smoothness conditions a representation of solutions of (***) in terms of pseudo-analytic functions is obtained which generalizes the Weierstrass representation of solutions of the minimal-surface equations by analytic functions. This theorem and theorems concerning isolated singularities and the generalization of Bernstein's theorem on entire solutions of the minimal surface equation are given detailed proofs in J. Rational Mech. Anal. 3, 767–787 (1954) [MR 16, 707]. Finally, results of the author concerning the existence of compressible subsonic flows past a profile are stated. The complete proof of this fundamental result on subsonic flows is given in Comm. Pure Appl. Math. 7, 441–504 (1954) [MR 16, 417]. *M. H. Protter.*

Vekua, I. N. On certain properties of solutions of a system of equations of elliptic type. Dokl. Akad. Nauk SSSR (N.S.) **98**, 181–184 (1954). (Russian)

This paper deals primarily with two properties of solutions of the equation (1) $w_{\bar{z}}=a(z)w+b(z)\bar{w}$ which is equivalent to a linear elliptic system for the real and imaginary parts of $w=u+iv$. (I) If a solution is given in a domain D, then it may be written in the form (2) $w(z)=e^{s(z)}f(z)$, where the function $f(z)$ is analytic, and the function $s(z)$ continuous. (II) Conversely, if an analytic function $f(z)$ is given in D, then one can find a function $s(z)$ such that $w(z)$ defined by (2) is a solution of (1). (For the sake of simplicity we assume here that D is bounded.)

Statements I and II have been proved by the reviewer [Proc. Nat. Acad. Sci. U. S. A. **37**, 42–47 (1951); MR **13**, 352; and the paper reviewed above] under the assumption that the coefficients a, b are bounded (for I) or bounded and Hölder-continuous (for II). In a previous paper [Mat. Sb. N.S. **31**(73), 217–314 (1952); MR **15**, 230] the author gave a proof of (I).

In order to find the desired function $s(z)$ in (I) one must solve the equation (3) $w_{\bar{z}}=A$, where $A(z)=-a-(\bar{w}/w)b$. Every solution of this equation differs from some particular solution by an analytic function, and a particular solution can be obtained by the formula

$$s(z)=-\frac{1}{\pi}\int\!\!\int_D\frac{A(\zeta)d\xi d\eta}{\zeta-z}$$

if the coefficients a, b are absolutely integrable. In this paper the author shows how I can be proved under more general conditions. He observes that if there exists an analytic function $\varphi(z)$ such that the product φA is integrable in the domain considered, then

$$s(z)=-\frac{1}{\pi\varphi(z)}\int\!\!\int_D\frac{\varphi(\zeta)A(\zeta)d\xi d\eta}{\zeta-z}$$

is a (generalized) solution of equation (3). Using this he proves I for the case of continuous a, b assuming that one can find analytic functions $\varphi(z)$, $\psi(z)$ such that the products φa and ψb will be absolutely integrable. This result is used in order to investigate the behaviour of solutions in the neighborhood of a regular or isolated singular point, and in order to derive a generalization of Liouville's Theorem. Then the author sketches a proof for II for the case of summable continuous coefficients a, b using linear integral equations.

In a footnote the author seems to object to the reviewer's previous statement [MR 15, 230] concerning his priority on I, but the substance of this objection is not clear to the reviewer. *L. Bers* (New York, N. Y.).

Vekua, I. N. The problem of reduction to canonical form of differential forms of elliptic type and the generalized Cauchy-Riemann system. Dokl. Akad. Nauk SSSR (N.S.) **100**, 197–200 (1955). (Russian)

This note contains a new proof of the following theorem. Let $ds^2=g_{11}dx^2+2g_{12}dxdy+g_{22}dy^2$ be a Riemannian metric defined in the $z=x+iy$ plane, with Hölder continuous coefficients and of bounded eccentricity. Then there exists a homeomorphism $w=u(x,y)+iv(x,y)$ of the plane conformal with respect to this metric. The proof does not use uniformization. It is based on writing the Beltrami system for w in the complex form (1) $w_{\bar{z}}=q(z)w_z$ where $|q|<q_0<1$. Setting $\omega=w_{\bar{z}}$, one considers the functional equation (2) $\omega=qS\omega+q$, where S is the singular integral operator

$$(3)\qquad S[\omega(z)]=-\frac{1}{\pi}\int\!\!\int\frac{\omega(\zeta)d\xi d\eta}{(\zeta-z)^2}.$$

Equation (2) is considered in L_p, for $p>2$ and $p-2$ sufficiently small. Calderón and Zygmund [Acta Math. **88**,

85–139 (1952); MR 14, 637] showed that the norm A_p of S in L_p is bounded. Since $A_2=1$, one has $\|qS\|\leq q_0A_p<1$ for $p-2$ sufficiently small. Then equation (2) has a solution. The author shows, assuming for the sake of simplicity that $q(z)$ has a compact carrier, that the solution ω of (2) is the $\bar{z}$-derivative of a univalent solution w of (1). The functions u and v are then isothermal coordinates for the metric ds^2.

The author also uses this method in order to prove the reviewer's theorem on the existence of univalent solutions of linear elliptic systems of the form (4) $u_x=av_x+bv_y$, $u_y=cv_x+dv_y$. He assumes that a, b, c, d are Hölder-continuous functions defined in the whole plane and that system (4) is uniformly elliptic. The reviewer's paper [Comm. Pure Appl. Math. 6, 513–526 (1953); MR 15, 431] dealt with differentiable coefficients, but with arbitrary domains and with non-uniformly elliptic equations. *L. Bers.*

Vekua, I. N. On a method of solution of boundary problems of partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) 101, 593–596 (1955). (Russian)

The author indicates an application of the Calderón-Zygmund inequality [Acta Math. 88, 85–139 (1952); MR 14, 637] to existence theorems for elliptic differential equation. The method is based on writing the solution of a homogeneous boundary-value problem for an elliptic equatio of order $2m$ in the form

$$u(x)=S_0\rho=\int G(x,\xi)\rho(\xi)d\xi,$$

where x and ξ are vectors in n-space and G the Green function of the same boundary value problem for $\Delta^m u=\rho$. The Calderón-Zygmund inequality is used to show that the operators

$$\frac{\partial^{2m}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}S_0$$

are bounded in L_p, $p>1$.

An application of this approach is sketched out for $m=1$, $n=2$ and the homogeneous Dirichlet problem in a disc, for the quasi-linear equation $au_{xx}+2bu_{xy}+cu_{yy}+d=0$. In complex form this equation may be written in the form

$$u_{z\bar{z}}+\operatorname{Re}[A(z,u,u_z)u_{zz}]+B(z,u,u_z)=0.$$

The functions $A(z,u,v)$, $B(z,u,v)$ are measurable for all real u and complex v and satisfy the following conditions (for some $p>2$).

(I) $|A(z,u,v)|\leq q(M)<1$ for $|u|+|v|\leq M$,

(II) $|A(z,u_1,v_1)-A(z,u_2,v_2)|<\text{const.}\times(|u_1-u_2|+|v_1-v_2|)$,

(III) $B(z,0,0)\in L_p$,

(IV) $|B(z,u_1,v_1)-B(z,u_2,v_2)|\leq B_0(z)|u_1-u_2|+B_1(z)|v_1-v_2|$,

where $B_0\in L_p$, $B_1\in L_p$. Theorem: The Dirichlet Problem for (1), $u=0$ on $|z|=1$, has a unique solution if the L_p norms of $B(z,0,0)$, $B_0(z)$ and $B_1(z)$ are sufficiently small. This solution has generalized second derivatives, the first derivatives satisfy a Hölder condition with exponent $1-(2/p)$.

For a uniformly elliptic linear equation

$$au_{xx}+2bu_{xy}+cu_{yy}+du_x+eu_y+fu=g,$$

with bounded measurable a, b, c and d, e, f, g in L_p, $p>2$, the homogeneous Dirichlet problem is solvable if either the L_p norms of d, e, f are sufficiently small or the solution is known to be unique. *L. Bers* (New York, N. Y.).

Heinz, Erhard. Über die Existenz einer Fläche konstanter mittlerer Krümmung bei vorgegebener Berandung. Math. Ann. 127, 258–287 (1954).

The Plateau problem [see, e.g., (*) R. Courant, Dirichlet's principle, Interscience, New York, 1950; MR 12, 90] is that of finding a minimal surface (mean curvature zero) spanning a given curve in 3-space. In this paper the author treats the more general (and more difficult) problem of the existence of a surface with constant mean curvature H which is homeomorphic to the disc and spans a given rectifiable Jordan curve Γ^* in space. In terms of isothermal coordinates $u+iv=w$ (in $|w|\leq 1$) this requires finding a vector $X(w)$ satisfying, in $|w|<1$,

$$X_{uu}+X_{vv}=2H(X_u\times X_v), \tag{1}$$
$$X_u^2=X_v^2,\quad X_u\cdot X_v=0, \tag{2}$$

and such that X is continuous in $|w|\leq 1$ and maps the boundary topologically onto Γ^*. One may assume that Γ^* lies within the unit sphere about the origin. In case Γ^* is a circle and $|H|\leq 1$ then either of the two spherical caps (radius $|H|^{-1}$) spanning Γ^* is a solution of the problem. Thus in general one expects two solutions for $|H|$ small and possibly none for $|H|$ large. The author proves the existence of one solution in case $|H|<H_1=8^{-1}(\sqrt{17}-1)$, which also lies in the closed unit sphere (corresponding, in case Γ^* is a circle, to the smaller spherical cap).

As with the Plateau problem the surface is obtained as a solution of a variational problem: Minimize

$$E[X]=\int_{|w|<1}(X_u^2+X_v^2)\,dudv+\frac{4}{3}H\int_{|w|<1}X\cdot X_u\times X_v\,dudv$$

among all vectors belonging to C_2 in $|w|<1$ which are continuous in $|w|\leq 1$, map $|w|=1$ in a monotone way onto Γ^* taking three given points into three given points, and satisfy $|X|\leq 1$. (Observe that the functional $E[X]$ need not be bounded from below if some condition like the last is not imposed.) The first step in solving the variational problem is to show that one can take as a minimizing sequence a sequence X_n^* of solutions of Euler's equation (1). To this end the author treats the boundary-value problem for (1) in $|w|<1$ with X continuous in $|w|\leq 1$ and (3) $X(e^{i\phi})=\tilde{X}(\phi)$ a given periodic continuous function, $0\leq\phi\leq 2\pi$ satisfying $|\tilde{X}|\leq 1$, and proves the following theorem. If $|H|<H_1$ the boundary-value problem admits a solution X^* with $|X^*|\leq 1$. Furthermore, X^* has finite Dirichlet integral and $E[X^*]\leq E[X]$ for any $X\neq X^*$ which agrees with X^* on the boundary and satisfies $|X|\leq 1$.

From the compactness of solutions of (1) (see (a), (b) below) it follows that a subsequence of the minimizing sequence X_n^* of solutions converge in $|w|\leq 1$ to a solution X^* of (1) which minimizes $E[X]$. In order to show that X^* satisfies (2) the author simply observes that the second integral in $E[X]$ is invariant under change of independent variable so that, since X^* minimizes E, it follows that the Dirichlet integral of $X^*\leq$ that of $X^*(\zeta(w))$, where $\zeta(w)$ is any C_2 homeomorphism of the closed unit disc onto itself with non-vanishing Jacobian. But it is known that any such function X^* then satisfies (2) (see (*), p. 107–115). Finally it is shown that X^* maps $|w|=1$ topologically onto Γ^*.

The major part of the paper is concerned with the proof of the theorem; this involves establishing strong a priori estimates for solutions X of (1) satisfying $|X|\leq 1$. To summarize, briefly, the author, using a mixture of potential-theoretic methods and bounds for Dirichlet integrals, de-

[I have learned that Hong's result was proved by Edgar Krahn at the end of his dissertation [Acta Comment. Univ. Tartuensis. Ser. A, 9 (1926), no 1]

rives: (a) bounds in the interior of the unit disc which yield compactness of such solutions; (b) an estimate for the modulus of continuity in $|w|\leqq 1$ of a solution satisfying (3) in terms of that of $\bar{X}(\phi)$; (c) estimates for $|X_u|$, $|X_v|$ in $|w|\leqq 1$ for solutions satisfying (3) with $\bar{X}$ in C_2. In (a) and (b), and in fact throughout the paper except in (c), it suffices that $|H|<\frac{1}{2}$. It is only in the proof of (c) that $|H|<H_1$ is required.

With the aid of the Leray-Schauder theory [Ann. Sci. Ecole Norm. Sup. (3) **51**, 45–78 (1934)] the existence of X^* of the theorem is then easily proved, using (a) and (c), assuming first that $\bar{X}$ is in C_2. Solutions X^* with given boundary values which are merely continuous are then obtained, with the aid of (a) and (b), as limits of solutions with C_2 boundary values. It should be remarked that by a further use of (a) and (b) follows the existence of a solution of the boundary value problem with $|H|=H_1$, and hence the existence of a surface spanning Γ^* with constant mean curvature H_1. The remainder of the theorem is easily established once it is known that X^* has finite Dirichlet integral. This the author proves by use of an expression due to J. Douglas [Trans. Amer. Math. Soc. **33**, 263–321 (1931), see pp. 307–311] for the Dirichlet integral of a harmonic function in terms of its boundary values.

The exposition is clear and complete, including, in an appendix, proofs of some known potential-theoretic results. Some questions are raised by the author's work: (i) Find a second solution surface X^{**} spanning Γ^* (which need not satisfy $|X^{**}|\leqq 1$); (ii) consider equations of the form: Laplacian of a vector $X=a$ function F of the first derivatives of X, where F is dominated by a constant C times the sum of squares of the first derivatives of X. The methods used in deriving (a)–(c) apply more generally to solutions X of such systems with $|X|\leqq 1$, provided C is sufficiently small. (Similar estimates under the same restriction have been recently derived by M. Nagumo in the paper reviewed below.) It would be of interest to know whether the restriction on C is necessary. *L. Nirenberg.*

Nagumo, Mitio. On principally linear elliptic differential equations of the second order. Osaka Math. J. **6**, 207–229 (1954).

The author considers the Dirichlet problem for a second-order elliptic equation of the form

$$(*)\qquad Lu-f(x_1,\cdots,x_n,u,u_{x_1},\cdots,u_{x_n})=0,$$

where L is a linear elliptic operator. A function $\bar{\omega}(x)$ is a quasi-supersolution of (*) in a domain D if for each point $x_0\in D$ there exists a neighborhood U of x_0 and a finite number of functions $\omega_\nu(x)\in C^2(U)$, $\nu=1,2,\cdots,n$, such that $\omega(x)=\min_\nu \omega_\nu(x)$, $x\in U$, and $L\omega_\nu\leqq f(x,\omega_\nu,\omega_{\nu x_1},\cdots,\omega_{\nu x_n})$. A quasi-subsolution $\underline{\omega}(x)$ is defined analogously. If the bounded domain D satisfies Poincaré's condition, if f satisfies several smoothness conditions one of which depends upon bounded quasi-supersolutions and quasi-subsolutions $\bar{\omega}$, $\underline{\omega}$, and if L is uniformly elliptic with Hölder-continuous coefficients (exponent 1) in $\bar{D}$, then the Dirichlet problem for (*) is solvable for boundary values $\beta(X)$ such that $\underline{\omega}(x)<\beta(x)<\bar{\omega}(x)$ in D.

The techniques follow those of Schauder [Math. Z. **38**, 257–282 (1934)] for linear equations. One of the conditions on $f(x,u,p)$ asserts that there should exist positive constants B and Γ so that

$$|f(x,u,p)|\leqq B|p|^2+\Gamma.$$

It is shown that this condition cannot be replaced by

$$|f(x,u,p)|<B|p|^\alpha+\Gamma \quad\text{for}\quad \alpha>2.$$

M. H. Protter (Berkeley, Calif.).

Hong, Imsik. On an inequality concerning the eigenvalue problem of membrane. Kōdai Math. Sem. Rep. **1954**, 113–114 (1954).

Let $\lambda_1<\lambda_2\leqq\cdots$ be the eigenvalues of $\Delta u+\lambda u=0$ in a plane domain D with area A, where $u=0$ on the boundary of D. Let $j\doteq 2.4048$ be the first zero of the Bessel function J_0. It is proved that $\lambda_2>(2\pi/A)j^2$, the fundamental eigenvalue of the circle of area $\frac{1}{2}A$. Any value $(2\pi/A)j^2+\epsilon$ can be attained for a domain D approximating two tangent circles each of area $\frac{1}{2}A$. *G. E. Forsythe.*

Dnestrovskiĭ, Yu. N. On the variation of eigenvalues with variation of the boundary of a region. Vestnik Moskov. Univ. **9**, no. 9, 61–74 (1954). (Russian)

The author considers the eigenvalue problem for the equation

$$\Delta u+\lambda u=0$$

in a region G, with the condition $u=0$ on the boundary Γ of G. Using a modification of a variational procedure due to Samarskiĭ [Thesis, Moscow State Univ., 1948; Dokl. Akad. Nauk SSSR (N.S.) **63**, 631–634 (1948); MR **10**, 458], he gives a method for computing approximations to the change of eigenvalues if the eigenfunctions are constrained to vanish on a small set adjacent to Γ or to nodal lines. Results obtained in this manner are shown to be superior in some cases to those given by perturbation theory. *R. Finn.*

Campbell, L. L., and Robinson, A. Mixed problems for hyperbolic partial differential equations. Proc. London Math. Soc. (3) **5**, 129–147 (1955).

For the hyperbolic equation

$$(1)\qquad \sum_{k=0}^{n}\sum_{l=0}^{k}a_{kl}(x,y)\frac{\partial^k z}{\partial x^{k-l}\partial y^l}=a_0(x,y)$$

the authors consider the initial and boundary-value problem defined by the relations

$$(2)\qquad z(0,y)=g_0(y),\ z_x(0,y)=g_1(y),\cdots,\left[\frac{\partial^{n-1}z}{\partial x^{n-1}}\right]_{x=0}=g_{n-1}(y)$$

on the interval I: $0\leqq y\leqq a$ on the y-axis, and

$$(3)\qquad \sum_{i=0}^{p_m}\sum_{j=0}^{i}\alpha^m{}_{ij}(x)\left[\frac{\partial^i z}{\partial x^{i-j}\partial y^j}\right]_{y=0}=\varphi_m(x)\quad (m=1,2,\cdots,k_0)$$

on the interval B: $0\leqq x\leqq c$ on the x-axis. It is assumed that none of the characteristic curves of (1) are tangent to B. The authors show that if k_0 is chosen to be the number of characteristics with positive inclination on B, if certain consistency conditions are satisfied, and if a certain determinant D, defined by the $\{a_{kl}\}$ and $\{\alpha^m{}_{ij}\}$, does not vanish on B, then there will be a region R bounded in part by I and B and a unique function $z(x,y)$ defined in R, such that $z(x,y)$ satisfies (1) in R, (2) on I, and (3) on B. The authors also give conditions under which a solution can be found interior to a closed polygon and satisfying prescribed conditions on the boundary.

The essential feature of the proof consists in observing that the integration of (1) is equivalent to the integration of an appropriate system of first-order equations, all of whose characteristic curves are characteristic curves of (1). This generalizes a method used by Robinson in a simpler

case [Aero. Res. Council, Rep. and Memo. no. **2265** (1950); J. London Math. Soc. **25**, 209–217 (1950); MR **12**, 338].

The authors point out that although several methods have been developed for the solution of initial-value problems for hyperbolic equations, little is known about conditions for solvability of initial and boundary-value problems, despite the evident physical interest of the latter case.

R. Finn (Los Angeles, Calif.).

Ladyženskaya, O. A. On the solution of the general problem of diffraction. Dokl. Akad. Nauk SSSR (N.S.) **96**, 433–436 (1954). (Russian)

Consider the hyperbolic equation

$$\rho(X)\frac{\partial^2 u}{\partial t^2}=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+a(X,t)u+f(X,t),$$

where the a_{ij} and ρ are piece-wise smooth functions having discontinuities of the first kind only on a surface F in X-space. For this equation the problem discussed is the following: (1) Cauchy data is given for $t=0$ in some domain Ω in X-space; (2) $u=0$ on the boundary of Ω for $0\leqq t\leqq t_1$ (or $\partial u/\partial N=0$); (3) on the surface F: $u_F{}^+=u_F{}^-$ and $(b\partial u/\partial N)_F{}^+=(b\partial u/\partial N)_F{}^-$, where $b(X)$ is a piece-wise constant function.

A similar problem is discussed for the system

$$\frac{\partial^2 u_i}{\partial t^2}=\sum_{k=1}^{3}\frac{\partial \tau_{ik}(u)}{\partial x_k}+f_i(X,t)\quad(i=1,2,3)$$

which arises in the theory of elasticity. In this case the piece-wise smoothness of the coefficients corresponds to the junction of different materials.

M. H. Protter.

Diaz, J. B., and Landshoff, Rolf. Solution for all values of the time, of initial value problems for the wave equation and for a system of equations in acoustics. J. Rational Mech. Anal. **4**, 503–515 (1955).

Les auteurs partent de la solution du problème de Cauchy sur $t=0$ pour l'équation

$$\left[\frac{\partial^2}{\partial t^2}-c^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]u=0,$$

sous la forme bien connue de V. Volterra [Acta Math. **18**, 161–232 (1894)] et la forme équivalente donnée par M. H. Martin [Bull. Amer. Math. Soc. **57**, 238–249 (1951); MR **13**, 244]. Ils en déduisent les deux formes explicites correspondantes de la solution du problème de Cauchy sur $t=0$ de l'équation

$$\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial t^2}-c^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]u=0$$

et du système

$$\frac{\partial u}{\partial t}+c^2\frac{\partial\lambda}{\partial t}=0,\quad\frac{\partial v}{\partial t}+c^2\frac{\partial\lambda}{\partial y}=0,\quad\frac{\partial\lambda}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

Des conditions précises pour la validité des formules résolutives sont signalées.

H. G. Garnir (Liège).

Jones, D. S. Note on Whitham's 'The propagation of weak spherical shocks in stars'. Proc. Cambridge Philos. Soc. **51**, 476–485 (1955).

The author establishes conditions for the validity of a method used by Whitham [Comm. Pure Appl. Math. **6**, 397–414 (1953); MR **15**, 751] in obtaining solutions to the differential equation

$$\chi_{tt}-A^2(\chi_{rr}+B\chi_r+C\chi)=0$$

The theory is also extended to the case of n independent variables.

M. H. Rogers (Urbana, Ill.).

Elianu, I. P. Invariants matriciels absolus pour les systèmes du type Laplace. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 847–852 (1954). (Romanian. Russian and French summaries)

Systems of Laplace's type, $z_{xy}+Az_x+Bz_y+C=0$, and their relative invariants H and K were investigated in an earlier paper [Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 155–196 (1953); MR **16**, 255]. The author now shows that HK^{-1} and $H^{-1}K$ are absolute invariants, and generates further absolute invariants by differentiation. A necessary condition for the equivalence of two systems of Laplace's type is the equivalence of their full systems of absolute invariants; and in the case of a single equation [Vrănceanu, Bull. Math. Soc. Roumaine Sci. **46**, 155–180 (1944); MR **7**, 450] this condition is also sufficient.

A. Erdélyi.

***Vahdati, A. N. Huygens' principle.** Tehran, 1955. v+54 pp.

Cet article a pour objet la majeure du principe de Huyghens pour les équations aux dérivées partielles linéaires hyperboliques du second ordre: l'état à l'instant t_2 d'un phénomène causé par une perturbation à l'instant t_0 peut être déduit de son état à un instant intermédiaire t_1. L'auteur utilise la méthode de résolution d'Hadamard et donne des relations intégrales vérifiées par ses solutions fondamentales.

Y. Fourès-Bruhat (Princeton, N. J.).

Olevskiĭ, M. N. On connections between solutions of the generalized wave equation and the generalized heat-conduction equation. Dokl. Akad. Nauk SSSR (N.S.) **101**, 21–24 (1955). (Russian)

The author states the following theorems. I) Let

$$u(x_1,\cdots,x_m,t;a,b;f)=u(P,t;a,b;f)$$

be a solution of the Cauchy problem:

$$(1)\quad A_Pu=\frac{\partial^2 u}{\partial t^2}+\frac{a}{t}\frac{\partial u}{\partial t}+bu\ \ (a\geqq 0),\ \ u|_{t=0}=f(P),\ \ \left.\frac{du}{dt}\right|_{t=0}=0,$$

where A_P is a linear differential-difference operator of the form

$$\sum_{\substack{l_1,\cdots,l_m=0\\ l_1+\cdots+l_m\leq l}}^{l}\ \sum_{q_1,\cdots,q_m=0}^{q}A_{q_1,\cdots,q_m}^{l_1,\cdots,l_m}(x_1,\cdots,x_m)\frac{\partial^{l_1+\cdots+l_m}W}{\partial x_1{}^{l_1},\cdots,\partial x_m{}^{l_m}},$$

$$W=u(x_1+\alpha_{q_1}^{l_1,\cdots,l_m},x_2+\beta_{q_2}^{l_1,\cdots,l_m},\cdots,x_m+\mu_{q_m}^{l_1,\cdots,l_m},$$

$\alpha_{q_1}^{l_1,\cdots,l_m},\cdots,\mu_{q_m}^{l_1,\cdots,l_m}$ are constants, and suitable smoothness is assumed of the coefficients. Then

$$(1_1)\quad u(P,t,a,b;f)=\int_0^{\pi/2}u(P,t\cos\varphi;a_1,b_1;f)$$
$$\times K(t,\varphi;a,b,a_1,b_1)d\varphi\quad(a>a_1\geqq 0),$$

where

$$K(t,\varphi;a,b,a_1,b_1)=C\bar{I}_{(a-a_1-2)/2}(t(b_1-b)^{1/2}\sin\varphi)$$
$$\times(\cos\varphi)^{a_1}(\sin\varphi)^{a-a_1-1}$$

$$C=\frac{2\Gamma(\frac{1}{2}(a+1))}{\Gamma(\frac{1}{2}(a_1+1))\Gamma(\frac{1}{2}(a-a_1))},\quad \bar{I}_\nu(z)=\sum_{s=0}^{\infty}\frac{\Gamma(\nu+1)(\frac{1}{2}z)^{2s}}{\Gamma(s+1)\Gamma(\nu+s+1)},$$

$$(1_2)\quad \mu(P,t;a,b;f)=Qt^{1-a}(t^{-1}\partial/\partial t)^n$$
$$\times[t^{2n+a-1}u(P,t;a+2n,b;f)]$$

$$(1+a)(3+a)\cdots(2n-1+a)Q=1$$

formula (1_2) holding under suitable differentiability assumptions.

II) Let $v(P, t; a, b; f)$ be a solution of the Cauchy problem

$$(2) \qquad A_P v = \frac{\partial^2 v}{\partial t^2} + a\alpha \operatorname{cth} \alpha t \frac{\partial v}{\partial t} + bv \quad (a \geqq 0),$$

$$v|_{t=0} = f(P), \quad \left.\frac{\partial v}{\partial t}\right|_{t=0} = 0.$$

Then

$$(2_1) \quad v(P, t; a, b; f) = \int_0^{\pi/2} v(P, \theta; a_1, b_1; f) L(t, \theta; a, a_1) d\theta,$$

$$(a > a_1 \geqq 0,\ b_1 = b - \tfrac{1}{4}\alpha^2(a^2 - a_1^2)),$$

where

$$L(t, \theta, a, a_1) = 2^{(a-a_1-2)/2} C\alpha (\operatorname{sh} \alpha t)^{1-a} (\operatorname{sh} \alpha\theta)^{a_1} (\operatorname{ch} \alpha t - \operatorname{ch} \alpha\theta)^{(a-a_1-2)/2},$$

$$(2_2) \quad v(P, t, a, b; f) = Q\left(\frac{\operatorname{sh} \alpha t}{\alpha}\right)^{1-a} \left(\frac{\alpha}{\operatorname{sh} \alpha t} \frac{\partial}{\partial t}\right)^n \times\left[\left(\frac{\operatorname{sh} \alpha t}{\alpha}\right)^{2n+a-1} v(P, t; a+2n, b_1; f)\right]$$

$$(b_1 = b + \alpha^2 n(a+n)).$$

III) Let $w(P, t; f)$ be à solution of the Cauchy problem

$$(3) \qquad A_P w = \frac{\partial w}{\partial t} \quad (t > 0), \qquad w|_{t=0} = f(P)$$

and let $u\ (P, t; a; f)$ be a solution of (1) for $b = 0$. Then

$$(3_1) \quad w(P, t; f) = \frac{2}{\Gamma(\frac{1}{2}(a+1))} \int_0^\infty e^{-\xi^2} \xi^a u(P, 2t^{1/2}\xi; a; f) d\xi.$$

An extension of (I) to the case $a < 0$ is also given. The proofs of these formulas are discussed with the remark that they follow by a direct verification.

The author indicates several applications of his theorems. First he notes that results of Weinstein [C. R. Acad. Sci. Paris **234**, 2584–2585 (1952); MR **14**, 176], of Diaz and Weinberger [Proc. Amer. Math. Soc. **4**, 703–715 (1953); MR **15**, 321] and of Kapilevič [Mat. Sb. N.S. **30**(72), 11–38 (1952); MR **13**, 750] appear as an immediate special case. Other applications consist in giving explicit solutions of a Cauchy problem for the wave equation and the heat equation in a space of constant negative curvature. Finally some integral and recurrence relations for Bessel functions are derived. *R. Finn* (Los Angeles, Calif.).

Magenes, Enrico. Problemi al contorno misti per l'equazione del calore. Rend. Sem. Mat. Univ. Padova **24**, 1–28 (1955).

Let D denote a domain in the (x_1, x_2) plane whose boundary FD has the parametric representation $x_i = x_i(t)$, $i = 1, 2$, $a_1 \leqq t \leqq a_2$. The functions $x_i(t)$ are assumed to belong to class C^2. Let F_1D be that part of FD determined by the range of t values $a_1 < t_1 < t < t_2 < a_2$, and use F_2D to denote $FD - F_1D$. Further, s_i denotes the set of points (x_1, x_2, y) with (x_1, x_2) on F_iD and $0 < y \leqq y_0$, and τ denotes the set of points (x_1, x_2, y) where (x_1, x_2) belongs to D and $0 < y \leqq y_0$. The author treats the following two types of mixed boundary-value problems for the heat equation

$$E(u) \equiv u_{x_1x_1} + u_{x_2x_2} - u_y = 0:$$

(1) $u = u(x_1, x_2, y)$ is a solution of $E(u) = 0$ for (x_1, x_2, y) interior to τ, on s_2 the normal derivative $du/dn = 0$, $u = 0$ on D, and (*) $u(x_1, x_2, y) = \mu(x_1, x_2, y)$ for (x_1, x_2, y) on s_1. (2) Conditions here are the same as those stated for problem (1) with the exception that condition (*), where u is assumed to take on the continuous assigned values μ on s_1, is replaced by the assumptions that μ belongs to L^2 and that u approaches μ in a certain limit-in-the-mean sense. Problems (1) and (2) are shown to have at least one solution, and uniqueness results are obtained for problem (2).

F. G. Dressel (Durham, N. C.).

Hirschman, I. I., Jr. Systems of partial differential equations which generalize the heat equation. Canad. J. Math. **5**, 118–128 (1953).

L'A. generalizza un risultato di Widder relativo all'equazione del calore, considerando il sistema a coefficienti costanti

$$(1) \qquad \frac{\partial u}{\partial h_l} = \sum_{i,j=1}^{m} \frac{\partial^2 u}{\partial x_i \partial x_j} a^l{}_{ij} \quad (l = 1, \cdots, n;\ a^l{}_{ij} = a^l{}_{ji}).$$

posto

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_m \end{bmatrix}, \quad H = \begin{bmatrix} h_1 \\ h_2 \\ \cdot \\ h_n \end{bmatrix}, \quad T = \begin{vmatrix} t_{11} & \cdots & t_{1m} \\ \cdot & \cdots & \cdot \\ t_{m1} & \cdots & t_{mm} \end{vmatrix},$$

$$X \varepsilon E_m, \quad H \varepsilon H_n, \quad T \varepsilon T_{m(m+1)/2},$$

si suive $T' > T$ se $T' - T$ è definita positiva e si stabilisce una rappresentazione dello spazio H_n su $T_{m(m+1)/2}$ ponendo

$$t_{ij} = \sum_{l=1}^{n} a^l{}_{ij} h_l \quad (H \to H_a\ \varepsilon\ T_{m(m+1)/2}).$$

L'A. dimostra allora che se T è definita positiva, se $u(X, H) \geqq 0$ soddisfa alle (1), allora risulta

$$(2) \quad u(X, H'') = \int_{E_m} k(X - Y, H_a'' - H_a') u(Y, H') dY,$$

purchè H', $H'' \varepsilon N$ (insieme aperto e convesso di H_n), $H_a'' > H_a'$, generalizza la nota soluzione fondamentale dell'equazione del calore. *L. Amerio* (Milano).

Rounds, Wellington, Jr. Solutions of the two-dimensional diffusion equations. Trans. Amer. Geophys. Union **36**, 395–405 (1955).

The form $4c_x = c_{zz} + A_z^{-1}c_z$ of the partial differential equation of diffusion, where $c(x, z)$ is the steady-state concentration in the region $x > 0$, $z > 0$, and A denotes a constant, is considered here. Particular solutions, written in terms of Bessel functions and exponential functions, are constructed from combinations of solutions with variables separated, using classical procedures. Types of boundary conditions which these solutions satisfy are then noted, and the solutions are interpreted as representing the concentration of diffusing gases in the atmosphere. Certain distributions of steady sources of the diffusing substance, over the plane $x = 0$, and certain distributions of wind velocity in the x direction are represented in the solutions. Concentrations are exhibited graphically.

R. V. Churchill.

Nardini, Renato. Completamento della soluzione di un problema al contorno della magneto-idrodinamica. Ann. Univ. Ferrara. Sez. VII. (N.S.) **2**, 17–33 (1953).

This is a continuation of previous work by the same author [Ann. Mat. Pura Appl. (4) **35**, 269–290 (1953); MR

16, 202]. It is concerned with solving

$$\frac{\partial^2 H(u,t)}{\partial t^2}=\frac{\partial^3 H(u,t)}{\partial t\partial u^2}+c\frac{\partial^2 H(u,t)}{\partial u^2}$$

in $0<u$, $t<\infty$ under the conditions

$$\begin{array}{lll} H\to 0, \quad \partial H/\partial t\to\Phi(u) & \text{as} & t\to+0\\ H\to G(t), & \text{as} & u\to+0\\ H\to 0, & \text{as} & u\to+\infty.\end{array}$$

In the previous paper the solution H_1, corresponding to $\Phi\equiv 0$ was found formally by a Laplace transform with respect to t. The same method is now used to find the solution H_2 corresponding to $G\equiv 0$. Finally sufficient conditions are found that the formal solution $H=H_1+H_2$ should be the unique solution of the general problem.

E. T. Copson (St. Andrews).

De Giorgi, Ennio. Un esempio di non-unicità della soluzione del problema di Cauchy, relativo ad una equazione differenziale lineare a derivate parziali di tipo parabolico. Rend. Mat. e Appl. (5) 14, 382–387 (1955).

The author constructs three functions $a(x,t)$, $b(x,t)$, $c(x,t)$ for which the following Cauchy problem does not have a unique solution in the set $0\leqq t\leqq 1$, $-\infty<x<\infty$,

$$(*)\qquad \begin{cases}\dfrac{\partial^8 w}{\partial t^8}=a\dfrac{\partial^4 w}{\partial x^4}+b\dfrac{\partial^2 w}{\partial x^2}+cw,\\[2mm] \dfrac{\partial^h w}{\partial t^h}=0, \quad t=0, \quad h=0,1,\cdots,7.\end{cases}$$

The constructed functions a, b, c as well as a nontrivial solution w of (*) have continuous partial derivatives of all orders.

F. G. Dressel (Durham, N. C.).

Levi, Beppo. On the general solution of the partial differential equation in two variables of order n, homogeneous with constant coefficients. Math. Notae 14, 50–63 (1954). (Spanish)

The author considers the equation

$$Lz\equiv\sum_0^n a_r\partial^n z/\partial x^{n-r}\partial y^r=0$$

with a_r constant, and the associated characteristic equation $f(\lambda)\equiv\sum_{r=0}^n a_r\lambda^{n-r}=0$. If $f(\lambda)\equiv\prod_k p_k(\lambda)$, where the polynomials p_k are pairwise relatively prime, then the general solution of $Lz=0$ can be expressed in the form $\sum_k P_k(x,y)$, where $P_k(x,y)$ satisfies an equation $L_k z=0$ whose characteristic equation is $p_k(\lambda)=0$. The method involves symbolic factorization of L and linear transformation of independent variables.

F. A. Ficken (Knoxville, Tenn.).

Difference Equations

Inoue, Masao. Discrete Neumann problem. J. Inst. Polytech. Osaka City Univ. Ser. A. 5, 101–109 (1954).

This paper treats the analog of the Neumann problem for discrete harmonic functions. A tabulation of the Neumann-type Green's function is given for a square region with 25 interior points.

R. J. Duffin (Pittsburgh, Pa.).

Duffin, R. J. Discrete potential theory. Duke Math. J. 20, 233–251 (1953).

The aim of the paper is to determine in what sense certain well-known properties of the (continuous) Laplacian differential operator Δ in three dimensions can be carried over to the corresponding difference operator, denoted by D, which operates on functions defined at "lattice" points whose rectangular coordinates are integers (functions u whose value at any lattice point is the mean value of its value at the six neighboring points are called "discrete harmonic," and the theory of the equation $Du=0$ is termed "discrete potential theory"). Recent work in potential theory shows that properties of Δ may be derived from properties of D, but in this paper known properties of Δ are used merely to suggest corresponding properties of D. It is known that the operator Δ may be treated by an operational calculus based on the Fourier transform; here it is shown that there is a corresponding operational calculus for D based on Fourier series. In particular, this calculus is used to show the existence of a (unique) "Green's function" g, which satisfies $Dg=0$ everywhere except at one point, where it has a unit source, $Dg=-1$, and vanishes at infinity. It is shown that the asymptotic behavior of g, at large distances from the source point, is like $1/4\pi k$ plus a term of order k^{-3}, where k is the distance from the source; the first term is precisely the potential of a unit source in continuous potential theory. To find this asymptotic expansion of g, the effect of a singularity of a function on the asymptotic behavior of its (multiple) Fourier transform at infinity is ascertained. The asymptotic property of g is used to prove an analogue of Gauss' mean-value theorem: a discrete harmonic function is, to a first approximation, the mean of its values at the lattice points of a concentric sphere, and this leads to an analogue of Harnack's inequality, which is used to give a new proof of the theorem (cf. Liouville's theorem in the continuous case) that a function which is positive and discrete harmonic everywhere must be a constant.

J. B. Diaz (College Park, Md.).

Saltzer, Charles. An abridged block method for the solution of the Dirichlet problem for the Laplace difference equation. J. Math. Phys. 32, 63–67 (1953).

The method presented is based upon the use of the explicit formula for the solution of the Dirichlet problem for the Laplace difference equation for a rectangular region. The procedure can be applied to any net which can be decomposed into a finite number of abutting or overlapping rectangles, and any finite net can be decomposed in such a way. The advantage of the method is that the number of equations which arise in the usual procedure is, in general, considerably reduced, and in some cases the number becomes so small that the system of equations can be solved directly. Consider the simplest case, that of two abutting rectangles L and R, and the Dirichlet difference boundary-value problem for the region $L+R$. Clearly, once the values of the solution have been determined for the common boundary points of L and R which are interior points of $L+R$, the solution of the Dirichlet problem for $L+R$ is then reduced to the solution of separate Dirichlet problems for L and R, which can be solved by means of the explicit solution formula for a rectangular region. Using this explicit formula for a rectangular region, and the fact that the Laplace difference equation must hold at each common boundary point of L and R which is an interior point of $L+R$, the author obtains a system of linear equations for the unknown values of the solution at these points, from which these values can be directly calculated first, in terms of the given Dirichlet data. This method is the finite-difference analogue of the method of B. Epstein [Quart.

Appl. Math. 6, 301–317 (1948); MR 10, 486] for the continuous case. The procedure can be easily extended to higher dimensions and more complicated meshes.
J. B. Diaz (College Park, Md.).

Ladyženskaya, O. A. On a method of approximate solution of the Lavrent'ev-Bicadze problem. Uspehi Mat. Nauk (N.S.) 9, no. 4(62), 187–189 (1954). (Russian)

For the equation $\partial^2u/\partial x^2+\theta(y)\partial^2u/\partial y^2=0$ with $\theta(y)=1$ for $y>0$, -1 for $y<0$, the author considers the finite-difference approximation to the Tricomi problem. Instead of taking a lattice over the entire domain, a lattice over the elliptic portion above is considered. The boundary conditions are: boundary values on the arc in the elliptic portion, and $\partial u/\partial l$ given on the x-axis, where $\partial/\partial l$ is the derivative in the characteristic direction. The derivative in this direction is known from the Tricomi data. The finite-difference problem is then solved in the elliptic portion of the domain. With this solution the result for the hyperbolic part follows at once. It is shown how the above finite-difference problem is solved; estimates for the difference between this solution and the solution of the differential equation are obtained.
M. H. Protter (Berkeley, Calif.).

Integral Equations, Equations in Infinitely Many Variables

Satô, Tokui. Sur l'application qui fait correspondre à une courbe une famille de courbes. Proc. Japan Acad. 31, 1–4 (1955).

Let $f(x)=\{f_j(x)\}$ and $K(x,t,u)=\{K_j(x,t,u_1,\cdots,u_n\}$ $(j=1,2,\cdots,n)$ be n-tuples of continuous and bounded functions for $0\leqq t\leqq x\leqq 1$, $|u_j|<\infty$. The main result of the paper is the following one: for given f the set of solutions u of the Volterra integral equation

$$(1)\qquad u(x)=f(x)+\int_0^x K(x,t,u(t))dt$$

is a continuum in the space (C) of n-tuples

$$u(x)=(u_1(x),\cdots,u_n(x))$$

continuous in $0\leqq x\leqq 1$. To prove this result mappings are considered which map an element of the product of (C) with the unit interval $0\leqq\lambda\leqq 1$ on a subset of (C) and which have certain specified continuity and compactness properties. [Cf. Hukuhara, same vol. 5–7 (1955); MR 16, 1140. Th. C of Aronszajn, Ann. of Math. (2) 43, 770–738 (1942) [MR 4, 100] also seems applicable to the proof of this theorem.]
E. H. Rothe (Ann Arbor, Mich.).

Fenyő, Stefan. Beitrag zur Theorie der linearen partiellen Integralgleichungen. Publ. Math. Debrecen 4, 98–103 (1955).

The author considers the partial integral equation

$$\varphi(x_1,x_2)-\lambda\int_0^1 A(x_1,x_2,y)\varphi(y,x_2)dy$$
$$-\mu\int_0^1 B(x_1,x_2,y)\varphi(x_2,y)dy$$
$$-\nu\int_0^1\int_0^1 C(x_1,x_2,y_1,y_2)\varphi(y_1,y_2)dy_1dy_2=f(x_1,x_2).$$

He gives conditions under which it has a unique solution $\varphi(x_1,x_2)$; in particular, this is so if

$$|\lambda|\cdot\|A\|+|\mu|\cdot\|B\|+|\nu|\cdot\|C\|<1,$$

the norm being the least upper bound, and all functions considered being bounded. A function playing the role of a resolvent kernel is introduced, and there is a brief discussion of the associated homogeneous equation.

Reviewer's note. The author seems to have misunderstood a remark in A. Salam's paper [Proc. Cambridge Philos. Soc. 49, 213–217 (1953); MR 14, 761], which indicates a possibility of multiple solutions, but does not explicitly allege that the multiplicity is genuine.
F. Smithies.

Evgrafov, M. A. Analog of Fredholm's theory for operators in spaces of analytic functions and a generalization of Poincaré's theorem on difference equations. Dokl. Akad. Nauk SSSR (N.S.) 101, 597–599 (1955). (Russian)

Let

$$K(z,\zeta)=\sum_{n=0}^\infty\sum_{k=0}^\infty a_{n,k}z^k\zeta^{-k-1}\quad(|z|<\rho<|\zeta|;\ c<\rho<d),$$

$$\limsup_{n\to\infty}\sum_{k=0}^\infty|a_{n,k}|r^{k-n}=\theta(r),\quad \theta=\sup\theta(r)\quad\text{for}\quad c<r<d.$$

Put

$$A(F)=(1/2\pi i)\int K(z,t)F(t)dt,$$

$$A^*(\phi)=(1/2\pi i)\int K(t,\zeta)\phi(t)dt,$$

integration along $|t|=r$, $c<r<d$. The author states the validity of the "Fredholm alternative" for the operators A and A^* (defined in the spaces of functions regular in $|z|<r$ and $|\zeta|>r$ respectively). He also states the existence of a resolvent $R=R(z,\zeta,\lambda)$ regular in z, ζ for $|z|<\rho<|\zeta|$, $c<\rho<d$ and meromorphic in $|\lambda|<1/\theta$ such that $(E+\lambda A)^{-1}$, $(E+\lambda A^*)^{-1}$ are integral operators of the same type as A and A^* with kernel R. At the poles λ_i of R, $(E+\lambda A)^{-1}$ does not exist, in this case the solutions of $(E+\lambda A)\phi=0$ can be expressed explicitly in terms of the residues of R. Applications: If $\phi_n(z)=z^nP(z)+\epsilon_n(z)\cdot z^{n+1}$, P, ϵ_n regular in $|z|<R$, $\epsilon_n\to 0$ as $n\to\infty$, then $\{\phi_n\}$ is complete in the space of analytic functions in a simply-connected domain not containing any zeros of $P(z)$. If ϕ is orthogonal to all ϕ_n, ϕ regular in $|z|>R_1$ $(R_1<R)$, then ϕ has a singularity at a zero of $P(z)$.
W. H. J. Fuchs (Ithaca, N. Y.).

Namias, V. Utilisation des propriétés formelles des fonctions impulsives δ_+ et δ_- pour la discussion de l'équation de Wiener-Hopf. Acad. Roy. Belg. Bull. Cl. Sci. (5) 41, 435–440 (1955).

In a previous note [same Bull. (5) 40, 787–790 (1954); MR 16, 261], Lafleur and the author discussed the use of the Dirac delta function in the solution of integral equations of the Wiener-Hopf type. The present note continues this discussion with some references to electric filter theory.
A. E. Heins (Pittsburgh, Pa.).

Lehner, Joseph, and Wing, G. Milton. On the spectrum of an unsymmetric operator arising in the transport theory of neutrons. Comm. Pure Appl. Math. 8, 217–234 (1955).

Let A denote the linear operator defined in a dense subset of the Hilbert space of measurable functions $f(x,\mu)$ such

that

$$\int_{-\infty}^{\infty} dx \int_{-1}^{1} |f(x, \mu)|^2 d\mu < \infty$$

by the equation

$$Af = -\mu \frac{\partial f}{\partial x} + \tfrac{1}{2} c \int_{-1}^{1} f(x, \mu') d\mu'.$$

This operator is not self-adjoint, and the authors prove that its spectrum is as follows: the point spectrum is a finite non-empty set of positive real numbers, the residual spectrum is empty, and the continuous spectrum is the entire half-plane $\Re(\lambda) \leqq 0$. In discussing the point spectrum, it is shown that if $\psi(x, \mu)$ is an eigenfunction with $\Re(\lambda) \geqq 0$, $\lambda \neq 0$, then the function

$$\varphi(x) = \int_{-1}^{1} \psi(x, \mu') d\mu'$$

satisfies the integral equation

$$2c^{-1}\varphi(x) = \int_{-\infty}^{\infty} E(\lambda|x-y|)\varphi(y) dy,$$

where $E(u) = \int_1^\infty t^{-1} e^{-ut} dt$. *F. Smithies.*

Warzée, J. L'équation intégrale de l'assombrissement du centre au bord du Soleil. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41**, 106–137 (1955).

A problem which has often been considered in the astrophysical literature is the following: Given the angular distribution of the emergent radiation $I(\mu)$, to determine the source function $B(\tau)$. The two functions are related by the integral equation

$$I(\mu) = \int_0^\infty e^{-\tau/\mu} B(\tau) \frac{d\tau}{\mu}.$$

In this paper the author suggests that in the astrophysical connections one can determine $B(\tau)$ with sufficient precision over the entire range of τ from the values of $I(\mu)$ for a discrete set of values for μ by assuming for $B(\tau)$ an expression of the form

$$B(\tau) = a_0 + a_1\tau - \sum_{n=1}^{\infty} \alpha_n e^{-n\tau} \qquad (\tau \geqq 0.2),$$

$$B(\tau) = a_0 + a_1\tau - \sum_{n=0}^{\infty} \beta_n (\tau \log \tau)^n \qquad (\tau \leqq 0.2),$$

where α_n and β_n are constants to be determined. In most cases two terms in the foregoing summations appear to suffice. The author verifies that in the case of a grey atmosphere for which $B(\tau)$ is known from a rigorous solution of the corresponding equation of transfer, his method predicts $B(\tau)$ (from a knowledge of $I(\mu)$) to a high degree of accuracy.
S. Chandrasekhar (Williams Bay, Wis.).

Meyer, J. A. Generalization of the Gross-transformation. An. Acad. Brasil. Ci. **26**, 375–380 (1954).

In interpreting the results of cosmic-ray measurements one has often to invert an integral equation of the form

$$N(x) = \int_\beta^\alpha f(\theta) I(x/\cos\theta) d\theta,$$

where $N(x)$ and $f(\theta)$ are known functions and α and β are assigned constants. The author points out that in most cases of practical interest the inversion is best accomplished by considering the Mellin transform of the equation. The explicit form of the solution is obtained in a number of elementary cases. *S. Chandrasekhar.*

Elliott, J. P. Milne's problem with a point-source. Proc. Roy. Soc. London. Ser. A. **228**, 424–433 (1955).

This paper deals with the solution of the integral equation

$$\rho(\mathbf{r}) = \frac{1}{4\pi} \iiint_{z'>0} \frac{\exp\{-|\mathbf{r}-\mathbf{r}'|\}}{|\mathbf{r}-\mathbf{r}'|^2} \times \{c\rho(\mathbf{r}') + q\delta(\mathbf{r}'-\mathbf{a})\} dx'dy'dz',$$

where c and q are constants and $\mathbf{a} = (0, 0, a)$ denotes a constant vector. For the Fourier transform

$$f(z; h_x, h_y) = \iint_{-\infty}^{+\infty} \rho(\mathbf{r}) \exp\{i(h_x x + h_y y)\} dxdy,$$

the author obtains the integral equation

$$f(z, h) = \tfrac{1}{2} c \int_0^\infty I(|z-z'|, h) \left\{ f(z', h) + \frac{q}{c} \delta(z'-a) \right\} dz',$$

where $h = (h_x^2 + h_y^2)^{1/2}$ and

$$I(\xi, h) = \int_1^\infty \frac{\exp\{-\xi t\}}{t} J_0\{h\xi(t^2-1)^{1/2}\} dt,$$

J_0 being a Bessel function. By using the Wiener-Hopf method the author obtains an explicit solution in the form of a definite integral over a known function for the case $a=0$. The solution for $a \neq 0$ is expressed in terms of the solution for $a=0$. The asymptotic form of the solution for $z \to \infty$ is obtained for $a=0$ and for the case $a \neq 0$ when $(c-1) \ll 1$.
S. Chandrasekhar (Williams Bay, Wis.).

Guirguis, G. K., and Hammad, A. Investigations on the height of the luminescent layer. Proc. Egyptian Acad. Sci. **9** (1953), 1–9 (1954).

This paper is concerned with the solution of the equation of transfer

$$\mu \frac{\partial I(r, \mu)}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial I(r, \mu)}{\partial \mu} = -\kappa\rho I(r, \mu) + \tfrac{1}{2}\kappa\rho \int_{-1}^{+1} I(r, \mu') d\mu',$$

where the notation is standard. In a certain approximation the solution can be written in the form

$$(*) \qquad \frac{1}{2} \int_{-1}^{+1} I(r, \mu) d\mu = A + B \int_r^R \frac{\kappa\rho dr}{r^2}.$$

[This solution is given in S. Chandrasekhar, Radiative transfer, Oxford, 1950, p. 367; MR **13**, 136.] The case considered by the authors corresponds to applying boundary conditions different from the usual for determining the constants of integration A and B in the solution (*). The application of the solution (*) to the particular physical problem considered by the authors is however novel.
S. Chandrasekhar (Williams Bay, Wis.).

Gurnee, Edward F. Solution of an integral equation arising in optical studies of oriented filaments. J. Appl. Phys. **26**, 918 (1955).

The author solves the integral equation

$$g(y) = 2R \int_y^1 (x^2 - y^2)^{-1/2} f(x) x dx$$

by reducing it to Abel's integral equation, and gives a few pairs of corresponding functions $f(x)$, $g(y)$. *A. Erdélyi.*

Grenander, Ulf. A contribution to the theory of Toeplitz matrices. Trans. Amer. Math. Soc. 79, 124–140 (1955).

Let m be a measure defined on a σ-algebra $\mathfrak{X}$ of subsets of an abstract set X. Let $\{\varphi_\nu\}$ be a complete orthonormal system in $L_2(X)$. Given a real-valued function f bounded and $(\mathfrak{X})$-measurable on X, the corresponding 'infinite Toeplitz matrix' $M(f)=\{m_{\nu\mu}(f); \nu, \mu=0, 1, \cdots\}$ is defined by $m_{\nu\mu}(f)=\int_X \varphi_\nu(x)\varphi_\mu^*(x)f(x)dm(x)$, where the asterisk denotes complex conjugation. This paper is concerned with the asymptotic behaviour of the eigenvalues $\lambda_\nu^{(n)}$ of the 'finite Toeplitz matrices' $M_n(f)$, which are the sections of $M(f)$, namely $M_n(f)=\{m_{\nu\mu}(f); \nu, \mu=0, 1, \cdots, n-1\}$. Let $D_n(t; f)=\{\text{number of } \lambda_\nu^{(n)}\leqq t\}/n$. If there is a distribution function $D(t; f)$ such that $D_n(t; f)\to D(t; f)$ as $n\to\infty$ at each point of continuity of $D(t; f)$, then $D(t; f)$ is called an 'asymptotic eigenvalue distribution'. A normalized measure on $\mathfrak{X}$ $(\mu(X)=1)$ is called a 'canonical distribution' if $D(t; f)=\mu\{x|f(x)\leqq t\}$. It was proved by Szegö [Math. Z. 6, 167–202 (1920); 9, 167–190 (1921)] that (Lebesgue measure)/(2π) is a canonical distribution for the classical Toeplitz matrices. The author shows that in a fairly general situation the problem of the existence of an asymptotic eigenvalue distribution can be reduced to the consideration of the first order moments $(\sum_{\nu=0}^{n-1} \lambda_\nu^{(n)})/n$ of the eigenvalues. Sufficient conditions for the existence of an asymptotic eigenvalue distribution and a canonical distribution are given. The theory is illustrated by examples involving particular orthonormal systems.
F. F. Bonsall.

Functional Analysis

★Kolmogorov, A. N., i Fomin, S. V. Èlementy teorii funkcii i funkcional'nogo analiza. I. Metričeskie i normirovannye prostranstva. [Elements of the theory of functions and of functional analysis. I. Metric and normed spaces.] Izdat. Moskov. Univ., Moscow, 1954. 154 pp. 5 rubles.

This admirable book presents within its brief compass most of the elements of the theory of metric spaces and normed linear spaces. It combines courses given by the authors for students of mathematics and of physics. While rigor is maintained at a high level, the needs of mathematical physicists have evidently been kept in view. The present volume is elementary in the sense that Lebesgue measure is neither assumed nor discussed. Later fascicules are promised, in which Lebesgue integration, Hilbert space, integral equations with symmetric kernel, and applications to computational mathematics will be dealt with. The scope of the present fascicule is indicated by the following summary.

Ch. I. Elements of the theory of sets. Intuitive set theory and operations on sets; cardinal numbers; functions.

Ch. II. Metric spaces. Elementary definitions and examples; topology associated with a metric; completeness and completion; compactness; Arzela's theorem; continuous curves. The elementary theorem is proved that if a mapping A of a complete metric space into itself has the property that $\rho(Ax, Ay)\leqq\alpha\rho(x, y)$ for all x, y and some α, $0<\alpha<1$, then A has a unique fixed point. This theorem is applied to a number of existence theorems in differential and integral equations. The discussion of continuous curves also contains much non-standard material.

Ch. III. Normed linear spaces. Elementary definitions and examples; convex sets; linear functionals; conjugate spaces; the Hahn-Banach theorem (proved only for real separable normed linear spaces); weak sequential convergence of elements and linear functionals; linear operators. The discussion of adjoint operators is particularly clear. An appendix gives a brief discussion of the 'generalized functions' of Sobolev (i. e., Schwartz's distributions), with emphasis on their applications in physics.

Ch. IV. Equations in linear operators. Spectra and resolvents for continuous linear operators in a complex Banach space; completely continuous operators; Fredholm's theorems.
E. Hewitt (Princeton, N. J.).

Sebastião e Silva, José. Su certe classi di spazi localmente convessi importanti per le applicazioni. Rend. Mat. e Appl. (5) 14, 388–410 (1955).

The author studies the processes of projective limit and inductive limit, applied to locally convex vector spaces, and defined in the usual way. He considers in particular the following two types of spaces: a space (M^*) is a projective limit of normed spaces S_n, the mapping of S_{n+1} into S_n being compact; a space (LN^*) is an inductive limit of an increasing sequence of normed spaces S_n, such that the injection of S_n into S_{n+1} is compact. He shows that these spaces are Hausdorff and Montel spaces; the dual of an (M^*) space is a (LN^*) space, and the dual of an (LN^*) space a (M^*) space; in an (LN^*) space, a set is closed if its intersection with each S_n is closed in S_n, and bounded sets are those contained in one of the S_n and bounded in that space. Finally, the author examines when a locally convex space E is such that a linear mapping from E into any locally convex space F, transforming convergent sequences into convergent sequences, is continuous; and he proves the interesting result that the space $\mathfrak{D}'$ of Schwartz distributions (over an R^n) has that property.
J. Dieudonné (Evanston, Ill.).

Ehrenpreis, Leon. Mean periodic functions. I. Varieties whose annihilator ideals are principal. Amer. J. Math. 77, 293–328 (1955).

Ce travail présente une généralisation partielle de résultats de L. Schwartz [Ann. of Math. (2) 48, 857–929 (1947); MR 9, 428]. Soit $\mathfrak{K}$ l'espace des fonctions entières dans C^n avec la topologie de la convergence uniforme sur les compacts et soit $\mathfrak{K}'$ son dual fort. D'après l'auteur, une variété de $\mathfrak{K}$ est un sous-espace vectoriel fermé de $\mathfrak{K}$, différent de $\{0\}$ et invariant par les translations. Une fonction $f\in\mathfrak{K}$ est dite moyenne-périodique, si la varieté engendrée par f est différente de $\mathfrak{K}$. On définit un produit de composition $S*T$ dans $\mathfrak{K}'$, en posant $(S*T)f=S_xT_zf(x+z)$ pour tout $f\in\mathfrak{K}$. Alors $\mathfrak{K}'$ devient une algèbre commutative et, si V est une variété de $\mathfrak{K}$, le sous-espace V' de $\mathfrak{K}'$ orthogonal à V est un idéal fermé de $\mathfrak{K}'$, dit l'idéal annihilateur de V. D'ailleurs, on définit un isomorphisme $\mathbf{F}$ de $\mathfrak{K}'$ sur l'espace des fonctions entières de type exponentiel, de façon que $\mathbf{F}(S*T)=\mathbf{F}(S)\mathbf{F}(T)$ (transformation de Fourier relative à $\mathfrak{K}'$).

L'auteur pose les problèmes suivants: 1) y a-t-il dans toute variété une exponentielle? 2) Toute variété V est-elle l'adhérence de l'ensemble des polynômes exponentiels (produits d'exponentielles par monômes) de V? L. Schwartz a donné des réponses affirmatives à ces questions dans le cas $n=1$. Le résultat fondamental de ce travail, concernant le cas $n>1$, est le théorème suivant: Si l'idéal annihilateur

de V est principal, V contient un polynôme exponentiel et tout $f \in V$ est la limite de polynômes exponentiels de V. L'auteur généralise encore ce résultat à d'autres espaces, en traduisant le problème, par la transformation de Fourier, en termes de la théorie des idéaux dans certains anneaux topologiques de fonctions entières.

Note du reviewer. L'espace $\mathfrak{K}'$ est isomorphe à l'espace $\mathfrak{A}$ des fonctions analytiques au point $(\infty, \cdots, \infty)$ de l'espace $\hat{C}^n$ de Osgood et nulles en ce point [cf. H.-G. Tillmann, Math. Z. **59**, 61–83 (1953); MR **15**, 211]. On peut définir dans $\mathfrak{A}$ le produit de composition et la transformation de Fourier par des intégrales de fonctions analytiques.

J. Sebastião e Silva (Lisbonne).

Ehrenpreis, Leon. Solution of some problems of division. II. Division by a punctual distribution. Amer. J. Math. **77**, 286–292 (1955).

L'auteur applique sa méthode, employée dans un travail précédent [même J. **76**, 883–903 (1954); MR **16**, 834], à la généralisation de résultats contenus dans ce travail au cas d'une équation de composition

$$(\alpha) \qquad D * T = S,$$

où D est n'importe quelle distribution de support ponctuel dans R^n. Pour toute distribution S d'ordre fini, on trouve une distribution T d'ordre fini vérifiant (α); si S appartient à la classe des fonctions indéfiniment dérivables, on peut choisir T dans la même classe. Il en découle, en particulier, que toute équation qui soit à la fois une équation différentielle et une équation aux différences finies, à coefficients constants, admet au moins une solution élémentaire au sens de Schwartz. *J. Sebastião e Silva* (Lisbonne).

Landesberg, Max. Über des Spektrum der Endomorphismen eines linearen Raumes. Math. Nachr. **13**, 1–8 (1955).

The greater part of this paper consists in results which are either well-known, or are obtained from such results by transposition. To get a complex vector space of dimension $\aleph$ (power of the continuum) in which there is an endomorphism with empty spectrum, the author considers the space of indefinitely differentiable functions; a simpler example would be the field $C(X)$ of rational functions over the complex field, with the endomorphism $f(X) \to Xf(X)$ for instance; the fact that $C(X)$ has a basis over C having the power of the continuum follows from the decomposition of a fraction in a linear combination of monomials of type X^h ($h \geq 0$) and $(X-a)^k$ ($k<0$). *J. Dieudonné.*

Bartle, R. G., Dunford, N., and Schwartz, J. Weak compactness and vector measures. Canad. J. Math. **7**, 289–305 (1955).

Let Σ be a σ-algebra of subsets of a set S and let ca(Σ) be the Banach space of countably additive, real or complex functions on Σ with $\|\lambda\|$ = total variation of λ over S. The authors give, first, two criteria for conditional weak compactness (cwc) of a subset K of ca(Σ). If μ is a function on Σ with values in a Banach space X, μ is called a vector measure if for each x^* in X^* the function $x^*\mu \in$ ca(Σ). By the first of the compactness criteria it is shown that if μ is a vector measure, then $\{x^*\mu \mid \|x^*\| \leq 1\}$ is cwc; by the other criterion there is a positive ν in ca(Σ) which dominates μ in a suitable sense.

These results are used to discuss a theory of integration of real-valued functions with respect to vector measures. The last section of the paper discusses representation of linear, compact, or weakly compact operators defined on $C(S)$, the space of continuous functions on a compact Hausdorff space, in terms of such integrals, and gives a number of properties of such operators. *M. M. Day.*

Shimoda, Isae. Notes on general analysis. IV. J. Gakugei. Tokushima Univ. Math. **5**, 1–7 (1954).

[For parts I–III see same J. **2**, 13–20 (1952); **3**, 12–15 (1953); **4**, 1–10 (1954); MR **14**, 766; **15**, 38, 801.] This paper deals with several topics in the theory of analytic functions defined on a complex Banach space, with values in another such space. The first section deals with measures of order for entire functions. If $M(r) = \sup_{\|x\|=r} \|f(x)\|$ and $M(r, x) = \sup_{|\alpha|=r} \|f(\alpha x)\|$, where $\|x\|=1$, let

$$\rho_1 = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}$$

and

$$\rho_2 = \sup_{\|x\|=1} \limsup_{r\to\infty} \frac{\log\log M(r, x)}{\log r}.$$

Then $\rho_2 \leq \rho_1$. The numbers ρ_1 and ρ_2 are expressible in terms of the behavior of the homogeneous polynomials occurring in the power series expansion of $f(x)$. Concerning isolated singular points there is an analogue of the Laurent series. The statement and proof of another theorem about singularities are not clear to the reviewer. Finally, there are some theorems about sums $\sum_{i=1}^n \|f_i(x)\|^p$, where each f_i is analytic. These are based on generalizations of Schwarz's lemma and the Hadamard three-circles theorem, and are related to results on functions of several complex variables, as given by S. Bochner and W. T. Martin [Several complex variables, Princeton, 1948; MR **10**, 366]. *A. E. Taylor.*

Shimoda, Isae. On isometric analytic functions in abstract spaces. Proc. Japan Acad. **30**, 718–720 (1954).

Suppose $f(x)$ is an entire function on a complex Banach space, with values in another such space. Suppose

$$\|f(x)\| \leq K\|x\|^m$$

for every x (m an integer and K fixed). Then $f(x)$ is a homogeneous polynomial of degree m. However, consider the following example: $f(x) = (x, x^2, \cdots, x^n, 0, \cdots)$, where x is a complex variable and the norm of the vector $(\xi_1, \xi_2, \cdots)$ is $\sup_i |\xi_i|$. Then $\|f(x)\| = |x|$ if $|x| \leq 1$ and $\|f(x)\| = |x|^n$ if $|x| > 1$. In this example f is not a homogeneous polynomial if $n > 1$. Thus the condition $\|f(x)\| = \|x\|^m$ for a restricted set of values of x does not imply that f is homogeneous of degree m. *A. E. Taylor.*

Polak, A. I. On the theory of linear functional equations. Uspehi Mat. Nauk **10**, no. 2(64), 175–177 (1955). (Russian)

The following result is obtained. Let C be a Banach space and let $\{L_n\}$ be a sequence of continuous linear operators in C such that if S is the unit sphere in C, then $\bigcap L_n(S)$ contains a non-void neighborhood of the origin. Let $g \in C$. If $\{L_n\}$ converges to a linear operator T uniformly on a non-void subset $G \subset C$ and if each equation $L_n(f) = g$ has at least one solution in G, then $f \in G$ is a solution of the equation $T(f) = g$ if and only if $f = \lim f_n$ where $L_n(f_n) = g$ ($n = 1, 2, \cdots$). This is established by first proving a similar result for mappings between metric spaces. The techniques are elementary. *R. G. Bartle* (Urbana, Ill.).

Fomin, S. V. On generalized eigenfunctions of dynamical systems. Uspehi Mat. Nauk (N.S.) 10, no. 1(63), 173–178 (1955). (Russian)

Let T be a C^∞ homeomorphism of a compact manifold Ω onto itself and let $S(\Omega)$ be the set of all infinitely differentiable functions on Ω. The author introduces the following definition: A linear functional ϕ_λ on $S(\Omega)$ is called a generalized eigenfunction, belonging to λ, of the dynamical system (Ω, T) if $\phi_\lambda(Uf)=e^{i\lambda}\phi_\lambda(f)$ for every $f \in S(\Omega)$, where Uf is defined by $Uf(p)=f(Tp)$. [For generalized functions, i.e. distributions, see L. Schwartz, Théorie des distributions, t. 1, 2, Hermann, Paris, 1950, 1951; MR 12, 31, 833.]

Let now Ω be the two-dimensional torus group and T one of its group automorphisms. Such an automorphism is known to be defined by an integer matrix $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ with determinant ± 1. The ordinary Lebesgue measure on Ω is invariant under T. It is known that if the characteristic roots of A have absolute value different from 1 then this dynamical system has a continuous spectrum. The author proves that in this case there exists a complete system of generalized eigenfunctions in the sense that $\phi(f)=0$ for each ϕ in the system only if $f\equiv 0$. The author remarks that the same considerations can be carried out for any dynamical system (Ω, T) with Ω an arbitrary compact commutative Lie group and T one of its group automorphisms.

Quite generally, if μ is an invariant measure on Ω, then to each ordinary eigenfunction $g_\lambda \in L_2(\mu)$ (satisfying $Ug_\lambda(p)=e^{i\lambda}g_\lambda(p)$) there corresponds a generalized eigenfunction $\phi_\lambda(f)=\int_\Omega \overline{g_\lambda(p)}f(p)\,d\mu$. Taking a special dynamical system on the torus known to have a pure point spectrum and hence having a complete system of ordinary eigenfunctions, the author verifies that every generalized eigenfunction in this case corresponds to some ordinary eigenfunction.

Y. N. Dowker (London).

Fichera, Gaetano. Formule di maggiorazione globale connesse ad una classe di trasformazioni lineari. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 371, 7 pp. (1953).

This note states without proof an inequality for a system of interrelated linear transformations in L_2-spaces, and applies it to a second-order partial differential operator of elliptic type, and to a related operator of parabolic type.

L. M. Graves (Chicago, Ill.).

Farinha, João. Sur la probabilité maximum d'accord de deux états. Rev. Fac. Ci. Univ. Coimbra 23, 21–22 (1954).

Schwarz' inequality holds strictly except for proportional vectors in Hilbert space.

I. E. Segal (Chicago, Ill.).

Svirskiĭ, I. V. On an estimate of the exactness of approximate methods of determining the oscillation frequencies. Izv. Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk 3, 59–86 (1953). (Russian)

In a Hilbert space let H be a self-adjoint operator with a discrete spectrum $\lambda_1\leqq\lambda_2\leqq\lambda_3\leqq\cdots$. The author notes that the variational method of Galerkin yields upper bounds $\lambda_{1r}, \lambda_{2r}, \cdots$, for the λ_i. He devotes the present paper to two methods for finding lower bounds, without referring to other literature on this subject.

In the first method, start with some $H_0\leqq H$ (meaning $(H_0u, u)\leqq(Hu, u)$ for all u), where one knows the eigenfunctions φ_i and eigenvalues λ_i^0 of H_0 explicitly. Define $H_1\leqq H_0$ so that

$$H_1f=\lambda_m^0f+\sum_{i=1}^{m-1}(\lambda_i^0-\lambda_m^0)(f, \varphi_i)\varphi_i.$$

Let $H_2=H-H_1$. For any $f_1, \cdots, f_n$ in $D(H_2)$, let $g_i=H_2f_i$ $(i=1, \cdots, n)$. The author then finds the operator H_3 which is least among all non-negative, self-adjoint operators A for which $Af_i=g_i$ $(i=1, \cdots, n)$, a result of apparent independent interest. Let $H_4=H_1+H_3$. It is shown that $H_4\leqq H$, so that $\lambda_i^{(4)}\leqq\lambda_i$ $(i=1, 2, \cdots)$. Moreover, the first $m+n-1$ eigenvectors of H_4 are in the subspace spanned by $\varphi_1, \cdots, \varphi_m$, $g_1, \cdots, g_n$, and can hence be computed explicitly as a finite matrix problem.

The second method leads by a long argument to the inequalities $(q=1, \cdots, m)$

$$(*)\qquad 0\leqq\lambda_{qr}-\lambda_q\leqq A_q^2(\lambda_{m+1}-2\sum_{i=1}^{m}A_i-\lambda_{qr}+A_q)^{-1},$$

where $A_q^2=(H\varphi_q, H\varphi_q)-(H\varphi_q, \varphi_q)^2$, and where the φ_q are the normalized eigenfunctions of the Galerkin estimates, with $(H\varphi_q, \varphi_q)=\lambda_{qr}$. Inequalities (*) are stated to be valid only when the denominator is positive; to apply them a rough lower bound for λ_{m+1} is needed.

The methods are illustrated numerically by three problems for the vibrating string or the clamped plate with physical properties varying in space. The absence of any stated theorem leaves the reviewer wondering if he has missed any assumptions.

G. E. Forsythe.

Vorob'ëv, Yu. V. Application of operator orthogonal polynomials to the solution of nonhomogeneous linear equations. Uspehi Mat. Nauk (N.S.) 10, no. 1(63), 89–96 (1955). (Russian)

Let A be a bounded self-adjoint operator defined on a Hilbert space H, and let $H_n=H_n(z_0)$ be the subspace spanned by $z_0, \cdots, A^{n-1}z_0$. Let B_n be the operator defined on H_n, with $z_k=B_n^kz_0$ $(k=0, \cdots, n-1)$, $\bar z_n=B_n^nz_0$, where $\bar z_n$ is the projection of z_n into H_n. The eigenvalues of B_n are roots of orthogonal polynomials $P_n(\lambda)$ treated by the author [Uspehi Mat. Nauk (N.S.) 9, no. 1(59), 83–90 (1954); MR 16, 146], and by W. Karush [Proc. Amer. Math. Soc. 3, 839–851 (1952); MR 14, 1127]. In the present paper the author considers how well B_n will approximate A for solving a linear equation (*) $Ax=f$.

Assume that the points $\{Q(A)z_0\}$ are dense in H, where Q ranges over all polynomials. Theorem 1: $B_n\to A$ strongly. Now let f_n be the projection of f into H_n; define $x_n=B_n^{-1}f_n$. Theorem 2: If A is also positive definite, then x_n converges strongly to the solution x_* of (*). The author shows how to obtain x_n constructively in terms of P_n, and notes that x_n minimizes $(Ax, x)-2(f, x)$ in H_n. Let $\eta_n=x_n-x_*$. The author gives an upper bound for $(A\eta_n, \eta_n)$ in terms of the maximum modulus of the nth Chebyshev polynomial. Theorem 2 is extended to certain indefinite operators with bounded inverses. Theorem 2 was proved by R. M. Hayes [Contributions to the solution of systems of linear equations and the determination of eigenvalues, Nat. Bur. Standards, Washington, D. C., 1954, pp. 71–103; MR 16, 597]. The estimate for $(A\eta_n, \eta_n)$ was given by the reviewer on p. 317 of Bull. Amer. Math. Soc. 59, 299–329 (1953); MR 15, 65.

G. E. Forsythe (New York, N. Y.).

Yood, Bertram. Multiplicative semi-groups of continuous functions on a compact space. Duke Math. J. 22, 383–392 (1955).

Let $C(X)$ be the Banach algebra of all continuous functions on a compact Hausdorff space X. This algebra may be considered as a multiplicative semi-group. A subset I of a semi-group B in $C(X)$ is a multiplicative ideal if $f \in I$ and $g \in B$ implies that $fg \in I$. An ideal I is an S-ideal if for any two elements f and g in I there is a sequence $\{e_k\}$ in I such that $e_k f \to f$ and $e_k g \to g$. A necessary and sufficient condition on B is given for the sets of the form $\{f \in B \mid f(t)=0, t \in K\}$, for some closed set K in X, to be all distinct and all the closed S-ideals in B. Several consequences are given.

D. C. Kleinecke (Albuquerque, N. M.).

***Kakutani, Shizuo. Rings of analytic functions.** Lectures on functions of a complex variable, pp. 71–83. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Let $A(D)$ [$B(D)$] denote the ring of all [all bounded] analytic functions on a schlicht domain D. In all discussion of $B(D)$, it is assumed in addition that for every point on the boundary of D, there is an f in $B(D)$ having a non-removable singularity at this point. It is shown that if for two such domains D, D', $A(D)$ [$B(D)$] and $A(D')$ [$B(D')$] are isomorphic as algebras over the complex field K, then D and D' are conformally equivalent. This theorem was proved first for $A(D)$ by L. Bers (without assuming that scalars are preserved) [Bull. Amer. Math. Soc. 54, 311–315 (1948); MR 9, 575], and for $B(D)$ by C. Chevalley and the author in 1942 (unpublished). Recently a simpler proof was given by W. Rudin [Trans. Amer. Math. Soc. 78, 333–342 (1955); MR 16, 685].

For each such ring, a maximal ideal is of type I or II according as it consists of all functions vanishing at a fixed point of D, or not. In each case, to prove the theorem cited above, it suffices to be able to distinguish between these types of maximal ideals in a purely algebraic way.

For $A(D)$, the maximal ideals of each type are characterized (the characterization of those of type II coincides with that given by the reviewer [Pacific J. Math. 2, 179–184 (1952); MR 13, 954] for the special case $D=K$). In particular, a maximal ideal M is of type I if and only if M is principal or $A(D)/M$ is isomorphic as an algebra to K. In case M is of type II, $A(D)/M$ is infinite-dimensional (as an algebra) over K. [These last two characterizations were first obtained by O. Helmer [Duke Math. J. 6, 345–356 (1940); MR 1, 307] and O. F. G. Schilling [Bull, Amer. Math. Soc. 52, 945–963 (1946); MR 8, 454] for the special case $D=K$.] The problem of determining the structure of $A(D)/M$, for M of type II, is left open. [In the paper cited above, the reviewer showed that, when $D=K$, $A(D)/M$ is always isomorphic as a ring (scalars need not be preserved) to K. In view of the author's Lemmas 2.1 and 2.2, the proof given therein is valid for arbitrary domains D.]

For the ring $B(D)$, where D is finitely connected, a maximal ideal M is of type I if and only if M is principal. For arbitrary domains (subject to the restriction cited in the first paragraph), a more complicated characterization (credited to C. Chevalley) is given. Several other possible characterizations of maximal ideals of Type I are conjectured. The structure of maximal ideals of Type II remains unknown.

M. Henriksen (Lafayette, Ind.).

Singer, I. M., and Wermer, J. Derivations on commutative normed algebras. Math. Ann. 129, 260–264 (1955).

A derivation D on an algebra A is a linear endomorphism with the property: $a,b \in A$: $D(ab)=D(a)b+aD(b)$. Theorem 1 is the basic result: By an argument involving Liouville's theorem on bounded entire functions, it is shown that every bounded derivation of a commutative Banach algebra maps into the radical. The remainder of the paper is devoted to applications of the above result. In particular, if a, b are bounded operators on a Banach space and $ab-ba$ lies in the uniformly closed algebra generated by a and 1, then $ab-ba$ is a generalized nilpotent. This generalizes a theorem of Wielandt [Math. Ann. 121, 21 (1949); MR 11, 38]. It is also shown that the algebra of all infinitely differentiable complex-valued functions on an interval cannot be suitably normed to obtain a Banach algebra. The paper closes with a discussion of derivations whose images lie in on extension of the algebra.

E. L. Griffin, Jr. (Ann Arbor, Mich.).

Vidav, Ivan. Über eine Vermutung von Kaplansky. Math. Z. 62, 330 (1955).

The conjecture is: if in a Banach algebra $ab-ba$ commutes with a, then $ab-ba$ is generalized nilpotent. Putnam [Proc. Amer. Math. Soc. 5, 929–931 (1954); MR 16, 490] proved this in a C^*-algebra, with the stronger hypothesis that $ab-ba$ commutes with both a and b. The author proves this result in any Banach algebra. He uses an exponential device akin to that employed by Singer and Wermer in the paper reviewed above. If we write $'$ for the inner derivation by b, the present hypothesis says that a' commutes with a and that $a''=0$. The Singer-Wermer argument works more generally if a commutes with all its derivatives. [To complete the record, I would like to record another proof of Vidav's theorem. The element $ab-ba$ lies in the center of the Banach algebra B generated by a and b. Hence $ab-ba$ is a scalar in any primitive image of B. By the original Wielandt theorem $ab-ba$ is 0 in any primitive image, i. e. it lies in the radical of B.]

I. Kaplansky (Chicago, Ill.).

Tomita, Minoru. Banach algebras generated by a bounded linear operator. Math. J. Okayama Univ. 4, 97–102 (1955).

Banach-algebraic preliminaries lead to the principal results, viz., Theorem 3: Let A be a bounded linear operator on a Banach space B. Necessary and sufficient conditions that a complex number z belong to S, the spectrum of A, is $\|r(A)\| \geq |r(z)|$ for every rational function r such that $r(A)$ exists; moreover, $\lim_{n\to\infty} \|r^n(A)\|^{1/n} = \sup_{z \in S} |r(z)|$. Theorem 4: If N is a Banach algebra of bounded linear operators in which $r(A)$'s are dense then $S(N)$, the spectrum of A in N, is a compact set such that $S(N)\setminus S$ is open. Conversely, such N exists uniquely, corresponding to any compact set T of complex numbers, $T \subset S$, $T\setminus S$ open.

J. G. Wendel (Ann Arbor, Mich.).

Misonou, Yosinao. On the direct product of W^*-algebras. Tôhoku Math. J. (2)6, 189–204 (1954).

Defining the direct product of two rings of operators as the ring (=weakly closed self-adjoint algebra of operators on a complex Hilbert space, which contains the identity) generated by the direct products of all pairs of operators in the respective rings, the author shows that algebraically isomorphic pairs of rings have algebraically isomorphic direct products. From this he deduces that the commutor of the direct product $(A\otimes B)'$ of two rings A and B is the

product $A'\otimes B'$ of the commutors, in the case of 'semifinite' rings (i. e. those with no constituent of type III). The proof is by reduction to the 'standard' case through the use of Hilbert algebras. Next the type question and the case of finite rings are studied. Among other results he obtains the following. The direct product of rings of type I (respectively, type II) is again of type I (resp. II). The fundamental group of the direct product of finite factors contains the fundamental groups of the factors as subgroups.

I. E. Segal (Chicago, Ill.).

Nakamura, Masahiro. On the direct product of finite factors. Tôhoku Math. J. (2)6, 205–207 (1954).

The author characterizes the direct product of rings [see the preceding review] in a special case as follows. A finite factor A is the direct product of finite factors B and C if and only if B and C commute element-wise, and B and C generate A.

I. E. Segal (Chicago, Ill.).

Turumaru, Takasi. On the direct-product of operator algebras. III. Tôhoku Math. J. (2)6, 208–211 (1954).

[For parts I and II see same J. (2) 4, 242–251 (1952); 5, 1–7 (1953); MR 14, 991; 15, 237.] The author shows that if the Banach space of all σ-weakly continuous linear functionals on a ring A [in the sense of Misonou; cf. the second preceding review] is denoted as A^*, then $(A\otimes B)^* = A^*\times_\alpha B^*$. Here "$\otimes$" denotes the direct product (cf. the cited review), while "$\times$" denotes the product in the sense of Schatten [A theory of cross spaces, Princeton, 1950; MR 12, 186] relative to a certain norm α. He then deduces the first result of Misonou (see the cited review), making use of a result of Kadison [Ann. of Math. (2) 54, 325–338 (1951); MR 13, 256].

I. E. Segal (Chicago, Ill.).

Takeda, Zirô. On the representations of operator algebras. II. Tôhoku Math. J. (2) 6, 212–219 (1954).

Suite d'un article antérieur [Proc. Japan Acad. 30, 299–304 (1954); MR 16, 146]. Soient A une B^*-algèbre, ρ_1 et ρ_2 des $*$-représentations fidèles de A dans des espaces hilbertiens, M_1 et M_2 les adhérences faibles de $\rho_1(A)$ et $\rho_2(A)$. L'auteur donne notamment des conditions, faisant intervenir les "états distingués" de A relativement à ρ_1 et ρ_2, pour que M_1 et M_2 soient algébriquement isomorphes. Application: si A_1 et B_1 (resp. A_2 et B_2) sont des anneaux d'opérateurs algébriquement isomorphes, les anneaux d'opérateurs $A_1\otimes A_2$ et $B_1\otimes B_2$ sont algébriquement isomorphes [Misonou, voir l'analyse du troisième article ci-dessus]. Si un facteur F est isomorphe, comme espace de Banach, au bidual d'une C^*-algèbre, F est de type I.

J. Dixmier (Paris).

Ogasawara, Tôzirô, and Yoshinaga, Kyôichi. Weakly completely continuous Banach *-algebras. J. Sci. Hiroshima Univ. Ser. A. 18, 15–36 (1954).

Let A be a Banach algebra (complex scalars) with an involution $x\to x^*$ and norm $\|x\|$. If A has a second "auxiliary norm" $|x|$ such that $k|xx^*|\geqq|x|^2$, k constant (completeness not assumed for $|x|$), then A is called an A^*-algebra [Rickart, Ann. of Math. (2) 51, 615–628 (1950); MR 11, 670]. In case $\|xx^*\|=\|x\|^2$, A is called a B^*-algebra. (In many results this condition can be replaced by $\|x\|^2\leqq k\|xx^*\|$, k constant). A Banach algebra A is said to be (weakly) completely continuous provided the right and left multiplications by any element of A are (weakly) completely continuous operators in A. Finally, A is said to be dual [Kaplansky, ibid. 49, 689–701 (1948); MR 10, 7] if for every closed right (left) ideal in A, $R(L(I))=I$ $(L(R(L))=I)$, where L and R denote the left and right annihilators respectively. The conditions of weak complete continuity and duality are shown to be equivalent for B^*-algebras. We mention here only a sampling of the variety of other results which the authors obtain for Banach *-algebras subjected to various of the above conditions. Many of the results generalize known ones for B^*-algebras. In all that follows A will be an A^*-algebra.

If the auxiliary norm of A satisfies the condition

$$\|ax\|\leqq c\|a\|\cdot|x|,$$

c constant, then any other auxiliary norm in A will be equivalent to $|x|$. Other conditions which give uniqueness up to an equivalence for the auxiliary norm are: (1) the socle (join of minimal ideals) is dense in A; (2) maximal commutative *-subalgebras are regular in the sense of Šilov [Trudy Mat. Inst. Steklov. 21 (1947); MR 9, 596]; (3) A is completely continuous; and (4) A is dual. If A is completely continuous, then it is dual if and only if the socle is dense in A and for every $x\in A$ the closure of xA contains x. If A is both completely continuous and dual, then every closed right ideal is the intersection of those regular right ideals which contain it. If A is dual, then there is a dual B^*-algebra $\mathfrak{A}$ in which A is dense and which is uniquely determined up to a *-isomorphism. $\mathfrak{A}$ will be completely continuous if and only if A is completely continuous. If A is an ideal in $\mathfrak{A}$, then A is said to be of the 1st kind. In order for A to be of the 1st kind it is necessary and sufficient that the norm $\|x\|_1=\sup\|xy\|$, $\|y\|=1$, satisfy a condition $\|x\|_1^2\leqq k\|xx^*\|_1$, k constant. If A is of the first kind, then any closed *-subalgebra B of A is a dual A^*-algebra of the 1st kind.

C. E. Rickart (New Haven, Conn.).

Ogasawara, Tôzirô, and Yoshinaga, Kyôichi. A characterization of dual B^*-algebras. J. Sci. Hiroshima Univ. Ser. A. 18, 179–182 (1954).

The main result in this paper is that a B^*-algebra A [see the preceding review for definitions] or a Banach algebra with an involution $x\to x^*$ satisfying a condition $\|x\|^2\leqq k\|xx^*\|$, k constant, is dual if and only if every self-adjoint element of A has a spectrum without cluster points other than zero. Also, if A is only an A^*-algebra, then this condition implies that A has a faithful *-representation by completely continuous operators in a Hilbert space.

C. E. Rickart.

Rutman, M. A. On certain operator equations in a partially ordered space which have application to the theory of stability according to Lyapunov. Dokl. Akad. Nauk SSSR (N.S.) 101, 217–220 (1955). (Russian)

Let E be a real linear partially ordered space with positive cone P. A Cauchy sequence $\{x_n\}$ in E is a sequence such that, for some $y\in P$ and every $\epsilon>0$ there exists N for which $-\epsilon y<x_m-x_n<\epsilon y$ whenever $m,\ n\geqq N$. Assume that E is complete, and that there exists a family of linear functionals f_α such that $x>0$ if and only if $f_\alpha(x)\geqq 0$, all α. Complexify E to $\bar{E}$ by defining $-y<x=x_1+ix_2<y$ to mean

$$-y<x_1\cos\theta+x_2\sin\theta<y,$$

all θ $(x_1, x_2\in E,\ y\in P)$. An operator S on E is of Volterra type if $x+\lambda Sx+\lambda^2S^2x+\cdots$ converges for every $x\in E$ and complex λ; monotone if $x>y$ implies $Sx>Sy$; majorized by T if $T\pm S$ are monotone; bounded, if for some scalar $c>0$, the relation $-y<x<y$ implies $-cy<Sx<cy$. Let S be an operator having a monotone Volterra majorant and let A be bounded and permutable with S. Without proof, existence

and uniqueness theorems are given for the solution of various equations of which the simplest is $y-SAy=x$, solution $y=-(2\pi i)^{-1}\oint_\gamma (I-\lambda S)^{-1}(A-\lambda I)^{-1}x\,d\lambda$, where γ is a suitable contour. From these results follow solutions to perturbation problems homologous to $y-SAy=x_0+Sx_1$. The application to Lyapunov theory is not spelled out.

J. G. Wendel (Ann Arbor, Mich.).

Calculus of Variations

De Giorgi, Ennio. Un nuovo teorema di esistenza relativo ad alcuni problemi variazionali. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **371**, 3 pp. (1953).

This note announces the existence of extreme values for any absolutely continuous set function whose domain is restricted to Borel measurable subsets E of a fixed bounded Borel measurable set D in r dimensions having $\phi(E)\leqq p>0$, where p is fixed and $\phi(E)$ is the Caccioppoli $(r-1)$-dimensional measure of the oriented frontier of E [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 3–11 (1952); MR **13**, 830]. *L. M. Graves* (Chicago, Ill.).

Bertolini, Fernando. Il problema del minimo per un classico funzionale. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **371**, 3 pp. (1953).

Let $f(x)$ be a real-valued function on a space X and $\{X_n\}$ an expanding sequence of sets filling X. If min $f(x)$ is attained in X_n and is independent of n for all n greater than some n', then min $f(x)$ is attained in X. This remark is applied to the fixed-endpoint problem: $\min_L \int_L \varphi(\rho)ds$, where $\varphi(\rho)$ is a positive continuous function, ρ is distance from the origin in cartesian space, and s is arc length along the curve L. *W. H. Fleming* (Lafayette, Ind.).

Pucci, Carlo. A proposito di un problema isoperimetrico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17** (1954), 345–346 (1955).

The problem is to find, among all domains D in space having a vertical plane of symmetry, having horizontal section similar to a given simply connected plane domain with rectifiable boundary, and having given volume V, the one whose surface has smallest Lebesgue area A. Existence and uniqueness are announced, as is the following necessary condition: $3LV=2\sigma A$, where σ and L denote, respectively, the area and the length of the perimeter of the maximal horizontal section of D. A more detailed exposition is to appear soon. *W. H. Fleming* (Lafayette, Ind.).

***Gillis, Paul P. Equations de Monge-Ampère et problèmes du calcul des variations.** III[e] Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 5–9. Fédération belge des Sociétés Scientifiques, Bruxelles.

Let D denote a simply connected domain in the xy-plane and $I(z)$ stand for the double integral that occurs frequently in problems of the calculus of variations

$$I(z)=\int\int_D [q^2r-2pqs+p^2t+6f(x,y,p,q)]dxdy.$$

Here $p=z_x$, $q=z_y$, $r=z_{xx}$, $s=z_{xy}$, $t=z_{yy}$, and f is assumed to be of class C^3. A function z will be said to belong to class $\mathcal{E}$ if it has the properties: (1) z belongs to class C^3 in D, (2) the inequalities

$$(*)\qquad (f_{pp}+t)(f_{qq}+r)-(f_{pq}-s)^2>0\quad (f_{pp}+t>0),$$

hold in D not only for the first and second derivatives p, q, r, s, t of z but also hold when these arguments in (*) are replaced by

$$p+\theta(\alpha_1-p),\quad q+\theta(\alpha_2-q),\quad r+\theta(\beta_1-r),$$
$$s+\theta(\beta_2-s),\quad t+\theta(\beta_3-t)\quad (0<\theta<1).$$

$\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ are values assumed by the first and second derivatives of any member of $\mathcal{E}$. The author proves that if the sequences of functions z_i, z_{ix}, z_{iy} converge uniformly to z, z_x, z_y, and if z_i, z belong to $\mathcal{E}$ then $\liminf_{i\to\infty} I(z_i)\geqq I(z)$.

F. G. Dressel (Durham, N. C.).

Fricke, A. Bemerkungen zu einer Variationsaufgabe. Elem. Math. **10**, 61–65 (1955).

This note discusses a geometrical problem which is equivalent to the problem of finding the curve of given length, with ends movable on the x-axis, which, when revolved about the x-axis, generates a surface enclosing the maximum area. The solution is expressed in terms of elliptic integrals. *L. M. Graves* (Chicago, Ill.).

Bellman, Richard, Glicksberg, Irving, and Gross, Oliver. On some variational problems occurring in the theory of dynamic programming. Rend. Circ. Mat. Palermo (2) **3** (1954), 363–397 (1955).

This paper concerns the problem of minimizing, by choice of $f(t)$, the cost of controlling a dynamical system governed by the equation

$$\frac{dx}{dt}=Ax+f(t),\quad x(0)=c,\quad 0\leqq t\leqq T,$$

where $x(t)$, $f(t)$, and c are vectors and A is a constant matrix. The main results have been announced previously [Proc. Nat. Acad. Sci. U. S. A. **39**, 298–301 (1953); MR **14**, 885]. The methods are special in that they depend on the specific (linear or quadratic) form of the cost functional. For example, existence and the first necessary condition are deduced by Hilbert space arguments. Many of the difficulties connected with problems of this type stem from inequality constraints restricting the nature of variations which can be made on $f(t)$. *W. H. Fleming.*

Theory of Probability

***Ackermann, W.-G. Einführung in die Wahrscheinlichkeitsrechnung.** S. Hirzel Verlag, Leipzig, 1955. x+185 pp. DM 28.00. on order

Elementary text book.

Mihoc, Gh. Definition of probability. Gaz. Mat. Fiz. Ser. A. **4**, 151–162 (1955). (Romanian)

van der Waerden, B. L., Pauli, W., and Rosin, S. Der Begriff der Wahrscheinlichkeit und seine Rolle in den Naturwissenschaften. Actes Soc. Helv. Sci. Nat. **132** (1952), 74–82; Diskussion 82–86 (1953).

Record of a symposium held 23 August 1952.

Jánossy, L. Remarks on the foundation of probability calculus. Acta Phys. Acad. Sci. Hungar. **4**, 333–349 (1955). (Russian summary)

A postulational treatment, taking statistical independence as an undefined relation between events. Probability as adduced here is defined only up to a monotone transforma-

tion, but there is one (and only one) such transformation for which the probability function has the usual properties. The existence of such a transformation is conjectured in this paper and proved in the paper reviewed below.

The author points out that this paper is close to an appendix of I. J. Good's book, Probability and the weighing of evidence [Griffin, London, 1950; MR 12, 837]. It is also reminiscent of a recent sequence of papers by H. Richter [Math. Ann. 125, 129–139 (1952); 223–234, 335–343; 126, 362–374 (1953); 128, 305–339 (1954); MR 14, 484; 15, 634; 16, 599]. *L. J. Savage* (Chicago, Ill.).

Aczél, J. A solution of some problems of K. Borsuk and L. Jánossy. Acta Phys. Acad. Sci. Hungar. 4, 351–362 (1955). (Russian summary)

Studies functional equations in 2 variables of which associativity, i.e.,

$$F[F(x, y), z] = F[x, F(y, z)], \quad \text{and} \quad F(x, y)z = F(xz, yz)$$

are typical. The principal object is to prove, and expand upon, a conjecture of L. Jánossy [see the paper reviewed above]. According to the author the paper is partly anticipated by a publication by I. J. Good [Probability and the weighing of evidence, Griffin, London, 1950; MR 12, 837]. *L. J. Savage* (Chicago, Ill.).

Thaer, C. Zum Petersburger Problem. Math. Naturwiss. Unterricht 6, 304–307 (1954).

The author seems to argue that the paradox is due to the essential unrepeatability of the game; and that hence medians, and not expected values, should be used. *A. Dvoretzky* (New York, N. Y.).

★Franckx, E. Calcul des probabilités. Formules globales de décomposition. III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 110–115. Fédération belge des Sociétés Scientifiques, Bruxelles.

Various elaborations of the elementary rules for calculating conditional probabilities. *H. P. McKean, Jr.*

Gini, Corrado. Sulla probabilità che x termini di una serie erratica sieno tutti crescenti (o non decrescenti) ovvero tutti decrescenti (o non crescenti) con applicazioni ai rapporti dei sessi nelle nascite umane in intervalli successivi e alle disposizioni dei sessi nelle fratellanze umane. Metron 17, no. 3–4, 1–41 (1955).

Reid, Walter P. Distribution of sizes of spheres in a solid from a study of slices of the solid. J. Math. Phys. 34, 95–102 (1955).

An infinite solid medium, in which spheres are embedded, is considered. A random plane slice intersects the sphere, in circles. Starting from the frequency distribution $F(R)$ of sphere radii in unit volume of the solid the author derives a formula for the distribution $f(r)$ of circle radii per unit area, and the inverse, $F(R)$ given $f(r)$. The corresponding formulae relating $F(R)$ to the length distribution, $\phi(y)$, of chords cut off on a random line are stated. The results are intended to apply when the distribution of spheres is at random in space. The author suggests that when this is not so they will still hold good if results from a number of spheres are averaged. The applicability to closely packed spheres is not considered explicitly; although the centres are not then distributed randomly, the results from a single slice may still be good enough if the packing is isotropic. *M. E. Wise* (Penarth).

Zoroa Terol, Procopio. On the independence of random variables. Trabajos Estadist. 5, 293–326 (1955). (Spanish. English summary)

Changes are rung on S. Bernstein's example of pairwise, but only pairwise, independent events. The possibility of transforming dependent sets of random variables to independent sets is explored. [Cf. I. E. Segal, Proc. Cambridge Philos. Soc. 34, 41–47 (1938); M. Rosenblatt, Ann. Math. Statist. 23, 470–472 (1952); MR 14, 189.] Measures of degree of dependence based on these transformations are suggested. The inadequacy of the correlation coefficient as such a measure, even for variables individually normally distributed, is stressed [cf. Fréchet, Rev. Inst. Internat. Statist. 18, 157–160 (1950); MR 12, 840]. The paper makes no reference to prior work, except for a casual mention of S. Bernstein. *L. J. Savage* (Chicago, Ill.).

Hoeffding, Wassily. The extrema of the expected value of a function of independent random variables. Ann. Math. Statist. 26, 268–275 (1955).

Let g_{ij} be given real functions of a real variable, c_{ij} given real numbers, A_j and B_j given extended real numbers, K a real function on R^n. Let C be the class of d.f.'s F on R^n for which $F(x_1, \cdots, x_n) = F_1(x_1) \cdots F_n(x_n)$ for some $F_1, \cdots, F_n$, with F_j having no variation outside the interval $[A_j, B_j]$ and with $\int g_{ij} dF_j = c_{ij}$ for $1 \leq i \leq k$, $1 \leq j \leq n$. Let C_m be the subset of F's in C for which each F_j has no variation outside of a set of m points. Under certain conditions on K and the g_{ij} and c_{ij} it is shown that $\int K dF$ has the same supremum (with respect to F) over C_{k+1} as over C and that this result cannot in general be improved by replacing C_{k+1} by C_k. The results generalize results of Tchebycheff and others. *J. Kiefer* (Ithaca, N. Y.).

Lancaster, H. O. Traces and cumulants of quadratic forms in normal variables. J. Roy. Statist. Soc. Ser. B. 16, 247–254 (1954).

The cumulants of a quadratic form in n independent standardized normally distributed random variables are identified with the traces of the powers of the matrix of the form. This identification leads to alternate proofs of well-known theorems, e.g., Cochran's theorem. A characterization of the normal distribution is presented by showing that only in a normal system is it possible to have non-trivial linear transformations from one set of independent random variables to another set of independent random variables. *M. Muller* (Ithaca, N. Y.).

Baxter, Glen. An analogue of the law of the iterated logarithm. Proc. Amer. Math. Soc. 6, 177–181 (1955).

We reproduce the first of the two theorems proved: Let X_{nk} $(k=1, \cdots, n; n=1, 2, \cdots)$ be independent random variables with X_{nk} assuming the values 1 and 0 with the probabilities λ/n and $1-\lambda/n$, respectively. Put

$$S_n = X_{n1} + \cdots + X_{nn};$$

then $\lim_{n\to\infty} S_n \log\log n/\log n = 1$ with probability 1. The situation here is actually much simpler than in the usual interated logarithm theorem and the result follows directly from the Borel-Cantelli lemma. *A. Dvoretzky.*

Hornby, Harold. Sur les fonctions aléatoires à symétrie hypersphérique avec composantes gaussiennes et stationnaires du second ordre. C. R. Acad. Sci. Paris 240, 2480–2482 (1955).

Let $x_1(t), \cdots, x_n(t)$ be independent, real, stationary, centered, Gaussian processes with the common infinitely

differentiable autocorrelation function

$$(*) \qquad \Psi(t)=E[x_j(t+s)x_j(s)] \quad (0<j\leqq n)$$

and set $R(t)^2=x_1(t)^2+\cdots+x_n(t)^2$. The author remarks that $R(t)$ is infinitely differentiable in the norm $\|x\|=[E(x^2)]^{1/2}$, (*) being real and even; computes the joint characteristic function for $R(t_1)^2, \cdots, R(t_m)^2$ $(0<t_1<\cdots<t_m)$ and the joint probability density for $R(t)^2$ and $(d/dt)R(t)^2$; and shows that $(d/dt)R(t)$ is strictly stationary, Gaussian, and independent of $R(t)$. Such processes have been studied by S. O. Rice [Bell System Tech. J. **23**, 282–332 (1944); **24**, 46–156 (1945); MR **6**, 89, 233].

Reviewer's note: the exponent in l. 2, p. 2482 should read $-\frac{1}{2}[\xi\Psi_0^{-1}+\eta^2(-4\Psi_0''\xi)^{-1}]$, the expression in l. 5, p. 2482 is the joint density for $R(t)^2$ and $(d/dt)R(t)$ and has to be divided by 4. *H. P. McKean, Jr.* (Princeton, N. J.).

Hornby, Harold. Sur une propriété invariante des fonctions aléatoires à symétrie hypersphérique avec composantes gaussiennes et stationnaires. C. R. Acad. Sci. Paris **241**, 353–355 (1955).

Continuing the paper reviewed above, the author computes the joint probability density for $R(t)^2$, $(d/dt)R(t)$, and $(d^2/dt^2)R(t)$ and shows that the expected rate at which maxima of $R(t)$ occur is

$$(*) \qquad (2\pi)^{-1}[(d^4/dt^4)\Psi(0)/(-d^2/dt^2)\Psi(0)]^{1/2}.$$

In addition, it is stated that the expected rate at which maxima of $(d/dt)R(t)$ occur is

$$(2\pi)^{-1}[-(d^6/dt^6)\Psi(0)/(d^4/dt^4)\Psi(0)]^{1/2}.$$

The calculation leading to (*) is based on a formula due to S. O. Rice [Bell System Tech. J. **24**, 46–156 (1945); MR **6**, 233]. *H. P. McKean, Jr.* (Princeton, N. J.).

Morgenstern, Dietrich. Über die Differentialgleichung des reinen Geburtsprozesses in der Wahrscheinlichkeitsrechnung. Math. Nachr. **13**, 57–58 (1955).

Choose positive numbers λ_n $(n\geqq 0)$ and set $\lambda_{-1}=0$. Then the solution of the system of differential equations,

$$P_n'(t)=\lambda_n P_n(t)+\lambda_{n-1}P_{n-1}(t) \quad (n\geqq 0, t>0),$$

subject to the initial conditions,

$$P_n(0+)=p_n \quad (n\geqq 0; 0\leqq p_n, \sum p_n=1),$$

is unique and $S(t)=\sum P_n(t)\leqq 1$ $(t>0)$. The necessary and sufficient condition that (*) $S(t)=1$ $(t>0)$ is that (**) $\sum\lambda_n^{-1}=+\infty$, a fact due to W. Feller [Trans. Amer. Math. Soc. **48**, 488–515 (1940); MR **2**, 101]. The author shows that (*)→(**). He believes that Feller's proof is not complete, but in this he is mistaken. *H. P. McKean, Jr.*

Cox, D. R. The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables. Proc. Cambridge Philos. Soc. **51**, 433–441 (1955).

From the author's abstract: "Certain stochastic processes with discrete states in continuous time can be converted into Markov processes by the well-known method of including supplementary variables. It is shown that the resulting integro-differential equations simplify considerably when some distributions associated with the process have rational Laplace transforms. The results justify the formal use of complex transition probabilities. Conditions under which it is likely to be possible to obtain a solution for arbitrary distributions are examined, and the results are related briefly to other methods of investigating these processes." A precise description of the processes considered is lengthy, but certain multiple-server queues are included. The mathematics is formal. The reviewer thinks p_{n+1} should be p_{n-1} in formula (8), and p_{n-1} should be p_{n+1} in formula (9). The function $p_0(\cdot)$ is used with two different meanings on pages 434 and 435.

T. E. Harris (Santa Monica, Calif.).

***Sarymsakov, T. A. Osnovy teorii processov Markova. [Elements of the theory of Markov processes.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 208 pp. 8.05 rubles.

This monograph deals almost exclusively with Markov processes with stationary transition probabilities. The level is roughly that of Fréchet's Recherches théoriques modernes sur la calcul des probabilités, 2ème livre [Gauthier-Villars, Paris, 1938] but it is not restricted to a finite number of states. Chapter 1 gives a general account and summary of contents. The approach is analytical and the relevant processes are not formally defined. Chapter 2 (61 pp.) deals with a finite number of states in the discrete-parameter case. The method is that of matrix theory, depending essentially on some results of O. Perron. The laws of large numbers and iterated logarithm are treated by Doeblin's method, while the central-limit theorem is treated by the method of characteristic functions, without recourse to the independence case. A useful section is Kolmogorov's local central-limit theorem [Izv. Akad. Nauk SSSR. Ser. Mat. **13**, 281–300 (1949); MR **11**, 119]. This part of the book gives a more condensed version of the material available in Fréchet's book.

Chapter 3 (51 pp.) deals with a denumerable number of states, discrete parameter. The treatment follows literally Kolmogorov's original version, followed by considerations of passage from finite to infinite matrices, said to afford a computational procedure. The latter material would be of more interest had a brief exposition of the basic matrix tools been included. In the treatment of the law of log log the method of Doeblin is not fully exploited and a uniformity condition is imposed entailing an exponentially fast convergence of the first return distribution. In a footnote on p. 136 this condition is replaced by one of which neither the meaning nor the truth is clear, the former because infinite determinants are involved and the latter because of the various major errors exhibited elsewhere (see below). [For this result and the central-limit theorem compare the reviewer's versions, Trans. Amer. Math. Soc. **76**, 397–419 (1954); MR **16**, 149.] In this chapter it is noticeable that the relevant material in Feller's book, though available in a Russian translation [cf. MR **16**, 722], is not even mentioned.

Chapter 4 (30 pp.) deals with a space which is a compact linear set with positive Lebesgue measure, discrete parameter. The method of integral equations is followed as a natural extension of that of matrices. A continuous transition density is assumed and on the whole the treatment is rather restricted in generality. Chapter 5 (27 pp.) deals with a finite or denumerable number of states, continuous parameter. Here the discussion is perfunctory, confined to some facile extensions of the discrete-parameter case. No mention is made of Doeblin's paper in Bull. Sci. Math. **62**, 21–32 (1938), nor the relevant work of Feller [Trans. Amer. Math. Soc. **48**, 488–515 (1940); **58**, 474 (1945); MR **2**, 101] and Doob [ibid. **52**, 37–64 (1942); **58**, 455–473 (1945); MR **4**, 17; **7**, 210], not to mention Lévy's [Ann. Sci. Ecole Norm. Sup. (3) **68**, 327–381 (1951); MR **13**, 959]. They

are all obviously too advanced for inclusion, but the author also seems unaware of the literature outside of Russia.

There are a number of alarming errors which seriously reflect on the author's scientific awareness, of which the following are examples. P. 66: proof of (8) is ridiculous. P. 110: the legitimacy of the passage to the limit in (11) is crucial but unnoticed. P. 134–135: an amazing proof of Theorem 3 (Doeblin's ratio limit theorem) since the existence of the limit is passed muster! If the author failed to understand Doeblin's paper [Bull. Sci. Math. **64**, 35–37 (1940); MR **1**, 343] which he cited—a pardonable failure indeed—he may try the reviewer's easier version [J. Res. Nat. Bur. Standards **50**, 203–208 (1953); MR **14**, 1099]. P. 177: the assertion in c. is transparently wrong: take $d=4$, $k=2$. This vitiates the proof of Lemma 3 on pp. 178–179.

It must be said that the author does not appear on solid ground in some of the things he discusses. Elsewhere this exposition is often lacking in elegance, balance and organization. While the book may still be useful as a handy collection of well-known and not-so-well-known results in an important branch of probability theory it is regrettably not on a par with the better books that have come out of the U.S.S.R.

K. L. Chung (Syracuse, N. Y.).

Austin, Donald G. On the existence of the derivative of Markoff transition probability functions. Proc. Nat. Acad. Sci. U. S. A. **41**, 224–226 (1955).

Let $p_{ij}(t)$ $(0<i, j)$ be stationary Markov transition probabilities, let $\lim_{t\to 0} p_{ii}(t)=1$, and let $\lim_{t\to 0} p_{ij}(t)=0$ $(i\neq j)$. Doob [Trans. Amer. Math. Soc. **52**, 37–64 (1942); MR **4**, 17] and Kolmogorov [Moskov. Gos. Univ. Uč. Zap. **148**, Mat. **4**, 53–59 (1951); MR **14**, 295] have shown that the $p_{ij}(t)$ are continuous on $[0, +\infty)$ and that the derivatives,

$$\lim_{t\to 0} t^{-1}(1-p_{ii}(t))=-p'_{ii}(0)\leqq +\infty$$

and

$$\lim_{t\to 0} t^{-1}p_{ij}(t)<+\infty,$$

exist.

The author shows that if, in addition, $p'_{ii}(0)>-\infty$ for some $i>0$, then the corresponding $p_{ij}(t)$ $(j>0)$ have continuous derivatives on $[0, +\infty)$ which satisfy

$$(1)\qquad \textstyle\sum_j p'_{ij}(t)=0,$$

$$(2)\qquad p'_{ij}(t+s)=\textstyle\sum_k p'_{ik}(t)p_{kj}(s),$$

$$(3)\qquad \textstyle\sum_j |p'_{ij}(t)|\leqq -2p'_{ii}(0).$$

The following misprints were noted: on p. 224, read $\geqq 0$ for $\leqq 0$ in l. 3; and, on p. 225, read $\Delta_{ij}(t_1, t_1+h_n)$ for $\Delta_{ij}(t_1+h_n)$ in l. 2, p_{jj} for p_{ij} in l. 4, D_- for D in l. 8, 4ϵ for ϵ in l. 10, and 3ϵ for 2ϵ in l. 22. *H. P. McKean, Jr.*

Ramakrishnan, Alladi. Inverse probability and evolutionary Markoff stochastic processes. Proc. Indian Acad. Sci. Sect. A. **41**, 145–153 (1955).

The probability distribution $\pi(t)$ of the random variable of a Markov process at time t satisfies an equation of the form $\partial\pi(t)/\partial t=F[\pi(t)]$. Formally, this equation can be used to find $\pi(t)$ given $\pi(s)$ as well for $t<s$ as for the usual case $t>s$, at least if t is not less than some minimum. This general principle is discussed and illustrated by examples.

J. L. Doob (Urbana, Ill.).

Nisida, Tosio. Note on Brownian motions with a parameter space R^n. Math. Japon. **3**, 85–91 (1954).

Simple geometrical interpretations are given to elementary calculations of variances and covariances of the random variables of a Brownian motion $\{X(A), A\in R^n\}$ with n-dimensional parameter space. A pair of independent motions defines a plane motion, and the author extends Lévy's weak Lipschitz condition [Processus stochastiques et mouvement brownien, Gauthier-Villars, Paris, 1948; MR **10**, 551] to the plane case: when $c>1$, but not when $c<1$, there is a positive ρ, depending on the sample function, such that, interpreting $X(A)$ as a plane vector,

$$|X(A)-X(B)|<c(2nr\log 1/r)^{1/2},$$

when $r=|A-B|<\rho$. Here A and B are to be restricted to a bounded set, a restriction omitted by the author.

J. L. Doob (Urbana, Ill.).

Pignedoli, Antonio. Su movimenti di tipo Browniano. J. Rational Mech. Anal. **4**, 579–593 (1955).

The author considers that neutrons, under certain circumstances, act like light diffusing particles in a heavy gas, and that the neutron energy at time t then becomes a random variable in a Markov process (stationary transition probabilities) with values between 0 and a certain barrier value. He solves the corresponding Fokker-Planck equation, whose coefficients are linear functions of the energy. The solution has the form $\sum_1^\infty \theta^{(i)}\Phi_i(x)\exp(-\alpha_i^2 t)$, where $\{\Phi_i, i\geqq 1\}$ is an orthogonal sequence on the energy interval, consisting of characteristic functions of a second-order linear differential equation, and $-\alpha_i^2$ is the characteristic value corresponding to Φ_i. *J. L. Doob* (Urbana, Ill.).

Whittle, P. The outcome of a stochastic epidemic—a note on Bailey's paper. Biometrika **42**, 116–122 (1955).

The author considers the following model of a stochastic epidemic [which is of a type due to Bartlett, J. Roy. Statist. Soc. Ser. B. **11**, 211–229 (1949); MR **11**, 672]. The population consists initially of n uninfected but susceptible individuals and a infected individuals. At any time t, if there are r susceptible and s infected cases, the probability of one new infection taking place in time Δt is $A(r)s\Delta t$ and the probability of one infected being removed from circulation is $Bs\Delta t$. The epidemic ends when either all infected have been removed or the whole population of $a+n$ individuals has become infected. If, when the epidemic ends the number of new infections is w (i.e. not counting the original a infections) it is said that the epidemic is of size w, and P_w denotes the probability of this event. Bailey [Biometrika **40**, 177–185 (1953); MR **14**, 1101] obtained a set of doubly recurrent relations for P_w, for the case that $A(r)=Ar$. In the present paper a set of singly recurrent equations is given for P_w. Further a closed expression is obtained for P_w for the special case $A(r)=$constant and this is used to derive simple criteria for the probability that the epidemic "takes" i.e. that w/n exceeds a predetermined fraction γ. *D. G. Chapman* (Seattle, Wash.).

Foster, F. G. A note on Bailey's and Whittle's treatment of a general stochastic epidemic. Biometrika **42**, 123–125 (1955).

The problem considered here is the same as that studied by Whittle [see the preceding review for an outline of the problem and also for references]. Here a set of singly recurrent relations for P_w is obtained by the use of simple probability argument, that is by treating the problem as one of random walk. A connection is also noted with the theory of fluctuations in coin-tossing [cf. Feller, An introduction to probability theory and its applications, v. I, Wiley, New York, 1950; MR **12**, 424]. *D. G. Chapman.*

Ammeter, Hans. Das Erneuerungsproblem und seine Erweiterung auf stochastische Prozesse. Mitt. Verein. Schweiz. Versich.-Math. **55**, 265–304 (1955).

An exposition of the renewal problem in demography. Comparison may be made with Joshi's paper [Publ. Inst. Statist. Univ. Paris **3**, 153–177 (1954); MR **16**, 731].

H. L. Seal (New York, N. Y.).

Kaiser, Ernest. Evolution récente de la théorie mathématique sur la distribution des revenus. Mitt. Verein. Schweiz. Versich.-Math. **55**, 305–335 (1955).

Reviews the mathematical theory of the distribution of incomes at varying attained ages x at different times t. Studies the relations between the distributions at successive epochs for different x and for the aggregate of x-values. No numerical examples. *H. L. Seal* (New York, N. Y.).

Elldin, Anders. On the congestion in gradings with random hunting. Ericsson Technics **11**, 33–94 (1955).

When telephone calls are assigned to trunks always in the same sequential order, those trunks first in order are busy more often than the last. Grading is a method of arranging the trunks, and the call assignments, in the interest of equal use of all trunks. The first cases studied were for sequential call assignment (hunting); here random assignment among idle trunks is considered. The particular case where there are three groups of calls, three groups of trunks, and each call group has access to one of the three pairs of trunk groups, is studied in detail for random call input in each group, stationary conditions, and rejection of calls when all trunks are busy. In the symmetric case, equal input in call groups and trunk groups equinumerous, the result is the same as in Erlang's "ideal grading" [Brockmeyer, Halstrøm, and Jensen, Trans. Danish Acad. Sci. **1948**, no. 2; MR **10**, 385]. A comparison is made with numerical results for the "loss" (the fraction of rejected calls) with sequential hunting given by C. Palm [Ericsson Technics **3**, 41–71 (1936)]; it appears that differences are small. Finally the identity with Erlang's result in the symmetric case is shown to be preserved when the number of trunk groups is arbitrary.

J. Riordan (New York, N. Y.).

McCombie, C. W. Fluctuation theory in physical measurements. Reports on Progress in Physics **16**, 266–320 (1953).

An elementary exposition of the applications of fluctuation theory to measurements involving suspended systems including radiation detectors such as thermocouples and bolometers. The topics treated include the effects of random forces (here the principal results are derived from the so-called Wiener-Khintchine formula which expresses the correlation between two random variables as a Fourier cosine transform of the corresponding power spectrum); Nyquist's formula for the power spectrum of the random voltage associated with a resistance, its extensions and generalizations; the possible attainment of the absolute limit of accuracy by feedback; and the optimum characteristics of an instrument used to follow a varying signal in the presence of noise. *S. Chandrasekhar* (Williams Bay, Wis.).

Kuznecov, P. I., Stratonovič, R. L., and Tihonov, V. I. The effect of electrical fluctuations on a vacuum-tube generator. Ž. Eksper. Teoret. Fiz. **28**, 509–523 (1955). (Russian)

This paper is the complete version of Dokl. Akad. Nauk SSSR (N.S.) **97**, 639–642 (1954); MR **16**, 725. An analysis is presented of the effect of noise on oscillatory processes. The assumption is made that the noise fluctuations are slow compared to the period of oscillation, but fast compared to the build-up time of the oscillations. An average over one cycle of the oscillation leads to generalized Langevin equations for amplitude and phase. These equations are used in the Einstein-Fokker [Fokker-Planck, note of reviewer] equation to obtain the probability density of the amplitude. A method leading to an infinite set of equations is suggested for the evaluation of the autocorrelation of the amplitude.

The effect of a noise voltage generator in the grid circuit of a vacuum tube oscillator is treated as an example. The probability distribution of the amplitude of the oscillations shows three possible states of oscillation: the undeveloped, the partially developed, and the fully developed oscillations. The variance of the probability distribution of the phase is derived. *H. A. Haus* (Cambridge, Mass.).

Jecklin, H., und Strickler, P. Warscheinlichkeitstheoretische Begründung mechanischer Ausgleichung und deren praktische Anwendung. Mitt. Verein. Schweiz. Versich.-Math. **54**, 125–161 (1954).

Given a set of linearly independent canonical functions $\phi_k(x)$ $(k=1, 2, \cdots, m)$, any function of the form $\sum_{k=1}^m \lambda_k\phi_k(x)$, where the λ_k are any real constants, will be termed a "standard" function. Given a set of observations $\cdots, y_{x-1}, y_x, y_{x+1}, \cdots$, the smoothed value $A(y_x)$ of y_x is defined as the corresponding ordinate of that standard function $y(x)$ for which $\sum_{j=-A}^{B} [y_{x+j}-y(x+j)]^2$ is a minimum, A and B being any chosen positive integers such that $A+B\geqq m$. It is shown that if the y_x are themselves values of a standard function, the smoothed values are identical with the observations, and also that the smoothed value can be expressed in the form $A(y_x)=\sum_{j=-A}^{B} f(x, j)y_{x+j}$, where $f(x, j)$ is a standard function of $x+j$ for any given x. A necessary and sufficient condition that $f(x, j)$ reduce to a function of j alone is that the $\phi_k(x)$ be solutions of a homogeneous linear difference equation of order m with constant (real) coefficients: in other words, that the form of the standard function be $\sum_i P_i(x)c_i^x$, where the c_i are the distinct roots of any chosen algebraic equation of degree m with real coefficients and $P_i(x)$ is an arbitrary polynomial of degree one less than the multiplicity of c_i. [The authors actually establish only the necessary (but not sufficient) condition that all the $\phi_k(x)$ satisfy a homogeneous linear difference equation of order $A+B$ with constant coefficients, and they discuss only the case in which $P_i(x)$ is always of degree zero for $c_i\neq 1$.] *T. N. E. Greville* (Washington, D. C.).

Mathematical Statistics

Chanda, K. C. On some moment properties when two polynomials have independent distributions. Calcutta Statist. Assoc. Bull. **6**, 40–44 (1955).

The paper consists of the following theorem: "Let $x_1, \cdots, x_n$ be n independent, but not necessarily identical, random variables with finite absolute moments of some positive order $\delta<1$. If, now, the linear function $y_1=\sum_{j=1}^n a_jx_j$ $(a_i\neq 0$ for $i=1, 2, \cdots, n)$ and the rth order polynomial $y_2=\sum c(p_1, \cdots, p_n)x_1^{p_1}\cdots x_n^{p_n}$, where $c(p_1, \cdots, p_n)$ are the coefficients and the summation is over all values of $p_1, \cdots, p_n\geqq 0$ subject to $\sum_{j=1}^n p_j=r$, the coefficient of x_i^r in y_2 being non-zero for all $i=1, \cdots, n$, are given to have inde-

pendent distributions, then moments of x_i ($i=1, \cdots, n$) of all finite order exist." *M. Muller* (Ithaca, N. Y.).

Ziaud-Din, M. On contributions to sampling distribution from symmetric functional point of view. Bull. Inst. Internat. Statist. 24, 2ème livraison, 207–226 (1954).

This is a review of the work done in the field of symmetric functions directed toward obtaining results of the moment of moments type in the sampling problem. Individual contributions are discussed briefly but the reviewer noted no omissions in the items included. There are a number of inaccuracies of statement not likely to bother a reader at all acquainted with the subject. At the end the author gives the values of the Fisher statistics k_9 and k_{10} in terms of power sums. *C. C. Craig* (Ann Arbor, Mich.).

Teghem, J. Sur la régression polynomiale, dans le cas où les valeurs de la variable indépendante sont en progression arithmétique lacunaire. Bull. Inst. Internat. Statist. 24, 2ème livraison, 109–121 (1954).

This is essentially a rearrangement of a previous article [Bull. Inst. Agronom. et Stations Rech. Gembloux 21, 160–170 (1953); MR 16, 77]. *R. L. Anderson.*

Wise, M. E. The effect of rounding off in samples. Statistica, den Haag 8, 169–173 (1954). (Dutch summary)

This note has to do with the sampling variance of means and variances calculated from grouped frequency tabulations with equal class intervals. It is assumed that the errors committed by replacing observed values of class marks are rectangularly distributed and that these errors are independent of the deviations of the observed values from the population mean. The first assumption is commonly made. The author notes that the second is more doubtful and engages in an inconclusive discussion of the point. In order to use strictly elementary methods the sample variance is taken about the population mean rather than about the sample mean which rather vitiates the result obtained.
C. C. Craig (Ann Arbor, Mich.).

Johnson, N. L. Systems of frequency curves derived from the first law of Laplace. Trabajos Estadist. 5, 283–291 (1955). (Spanish summary)

Let z be distributed according to the First Law of Laplace, $p(z)=\frac{1}{2}e^{-|z|}$. The author studies systems of distributions of chance variables y which are generated by translations of the form $z=\gamma+\delta f(y)$, where γ and $\delta>0$ are constants and $f(y)$ is a function of the following forms: $f(y)=\log y$, $\log y/(1-y)$, $\sinh^{-1} y$. Problems of fitting distributions belonging to these systems are discussed. *G. E. Noether.*

Tintner, Gerhard. The distribution of the variances of variate differences in the circular case. Metron 17, no. 3–4, 43–52 (1955).

The sample variance of the kth difference series of observations $X_1, X_2, \cdots, X_N$ is defined by $V_k=\sum_{i=1}^{N} (\Delta^k X_i)^2/N\binom{2k}{k}$, where $\Delta^k X_i$ is the kth finite difference of the item X_i, the differences computed in the circular fashion. The author derives the characteristic function of the distribution of V_k under the assumption that the $X_1, \cdots, X_N$ are normally and independently distributed with zero mean and the same variance. The cumulants and measures of skewness and kurtosis are then derived. The distribution of V_k is derived by the inversion formula from the characteristic function. The resulting distributions are suggested as approximations for the noncircular case also. For $N=7$, $\sigma^2=1$, $k=2$ and $N=8$, $\sigma^2=1$, $k=2$, comparison curves are drawn for the exact distribution and a normal approximation.
S. Kullback (Washington, D. C.).

Thompson, W. A., Jr. The ratio of variances in a variance components model. Ann. Math. Statist. 26, 325–329 (1955).

The author is concerned with the test of the ratio of two variances which arise in the mixed incomplete block model, where N observations y_{ij} for $i=1, 2, \cdots, u$ and $j=1, 2, \cdots, b$ are independent and normal for any given $t_1, t_2, \cdots, t_u$ with means $E(y_{ij}/t)=n_{ij}(t_i+b_j)$ and variances σ^2, n_{ij} is 1 or 0 according to the ith treatment does or does not occur in the jth block. Here the t's are assumed to be independent and normal with mean 0 and variance ϵ^2. The author adheres to the principle of invariance and finds the class of invariant statistics and obtains the joint distribution of these statistics, which yields a test which maximizes the minimum powers among all invariant tests for the hypothesis $\epsilon^2/\sigma^2\equiv\lambda<\lambda_0$ versus $\epsilon^2/\sigma^2\equiv\lambda>\lambda_1$. *T. Kitagawa* (Ames, Iowa).

Kitagawa, Tosio. Some contributions to the design of sample surveys. Sankhya 14, 317–362 (1955).

In Part I the author, following the usual practice in sample-survey theory of assuming stratum sizes, variances and costs to be known at least approximately, solves certain optimum allocation problems in multipurpose non-sequential sampling. E.g., he determines optimum allocation of a one-way stratified sample when the object is to estimate a set of linear combinations of stratum means in a two-way stratification system, the second stratification being applied after selection. In Part II the author determines the loss of efficiency due to the approximations in sizes, variances and costs. In Part III he treats various sequential cases, when one designs a sequence of sample surveys, such that the results of each survey are used for the improvement of the following surveys. The simplest of these cases is the following. By a preliminary sample, the cost of which is a certain fraction a of the given total cost C, unbiased estimates $\hat{\sigma}=(\hat{\sigma}_1, \cdots, \hat{\sigma}_k)$ of the stratum standard deviations $\sigma=(\sigma_1, \cdots, \sigma_k)$ are obtained. These are used for allocating a second sample among the k strata. The cost of this sample should be $(1-a)C$. Let $\bar{x}$ be the estimated population mean, $V(\bar{x})$ the variance of $\bar{x}$, and $V_\sigma(\bar{x})$ the variance of $\bar{x}$ in case σ were known (and $a=0$). The author defines a relative loss function $W_r(\sigma; a, C)$ as $V(\bar{x})/V_\sigma(\bar{x})$. Assuming that Ω is the set of all points σ such that every $\sigma_i>0$, and that $E[\sigma_i/\hat{\sigma}_i]=1+\gamma n_0^{-1}$, where n_0 is the size of the preliminary sample and γ is a constant not depending on n_0 or i, he determines the value of a corresponding to $\min_{0\leq a\leq 1} \sup_\Omega W_r(\sigma; a, C)$. *D. M. Sandelius.*

Kitagawa, Tosio. Empirical functions and interpenetrating sampling procedures. Mem. Fac. Sci. Kyûsyû Univ. A. 8, 109–152 (1954).

The author combines lines of thought from three previous papers [Bull. Math. Statist. 4, 15–21 (1950); 5, no. 3-4, 19–33 (1953); MR 14, 457; 15, 886; and the paper reviewed above]. In his Part I he considers a fixed function $g(t)$, $0\leq t\leq 1$, whose values at the points of a fixed finite set τ are estimated from observations, and he outlines procedures for estimating its maximum value and for testing it for periodicity. In Part II the fixed set τ is replaced by a set chosen at random, and he discusses the estimation of (generalized)

Fourier coefficients of $g(t)$. Part III deals with cases of "interpenetration", in which we consider initially two or more independent estimates of $g(t)$, and, after comparing them by a significance test, we choose one of them, or a combination of them, as our final estimate. The author's multifarious procedures and formulae are too elaborate for reproduction here, and in Part III his analysis in most cases would not suffice as a basis for computation of the requisite sampling distributions. *H. P. Mulholland.*

Lieberman, Gerald J., and Resnikoff, George J. Sampling plans for inspection by variables. J. Amer. Statist. Assoc. 50, 457–516 (1955).

The authors are interested in the theory of acceptance sampling when the characteristic considered is measurable, distributed normally, and the decision to accept or reject a lot is made on the basis of the percentage of defectives permitted in the lot, p^*. Three types of plans are given: (1) mean unknown, standard deviation known; (2) mean and standard deviation unknown; and (3) the same as case (2), but the average range usually in sub-groups of 5 replaces the sample standard deviation. Case (3) is new. The methods and results are similar to those of Bowker and Goode [Sampling inspection by variables, McGraw-Hill, New York, 1952], but emphasis has been placed on estimation of the present defective in the lot as well as the decision whether to accept or reject the lot. Upper and lower one-sided limits and two-sided limits are employed with fixed single sample size. Many well planned tables aid in the choice of a particular plan. The consequences of each plan are indicated in the graphs of the corresponding O.C. curves.
L. A. Aroian (Culver City, Calif.).

Bartlett, M. S. Approximate confidence intervals. III. A bias correction. Biometrika 42, 201–204 (1955).

The author corrects an error in his second paper on this topic [Biometrika 40, 306–317 (1953); MR 15, 544]. Several examples are worked out which check his results.
H. Chernoff (Stanford, Calif.).

Epstein, Benjamin. A sequential two sample life test. J. Franklin Inst. 260, 25–29 (1955).

A sequential procedure is devised for testing the hypothesis $\theta_1=\theta_2$ against the alternatives $\theta_1>\theta_2$, or $\theta_1<\theta_2$ when testing two lots of electron tubes having a life time prescribed by the exponential probability density function $\theta_i^{-1}e^{-x/\theta_i}$, $x>0$, $\theta_i>0$, $i=1, 2$. *L. A. Aroian.*

Cochran, William G. A test of a linear function of the deviations between observed and expected numbers. J. Amer. Statist. Assoc. 50, 377–397 (1955).

Let f_i be observed frequencies with expectations M_i (hypothesis H_0), which are known functions of k unknown parameters θ_j, let $\hat{\theta}_j$ be maximum-likelihood estimates of the θ_j based on the f_i, and let m_i be the values of the M_i corresponding to the $\hat{\theta}_j$. The author derives, for $k=1$ and $k=2$, an estimate $\hat{V}(L)$ of the variance of $L=\sum g_i(f_i-m_i)$ and suggests $L^2/\hat{V}(L)$ as a test of H_0, the g_i being chosen so that the test will be sensitive to the most plausible alternative to H_0. Several illustrations are given. *D. M. Sandelius.*

Fix, Evelyn, and Hodges, J. L., Jr. Significance probabilities of the Wilcoxon test. Ann. Math. Statist. 26, 301–312 (1955).

Tables are presented from which exact values of the Wilcoxon distribution may be obtained when the smaller sample size m does not exceed 12. For larger sample sizes one may apply the Edgeworth series which is given to terms of order $1/m^2$. Some examples are worked out which indicate the good accuracy which may be obtained with this approximation. *H. Chernoff* (Stanford, Calif.).

Patnaik, P. B. A test of significance of the standardised mean. Bull. Inst. Internat. Statist. 23, part II, 163–170 (1951).

The author claims to derive an optimal test for the hypothesis $\mu/\sigma=P_0$ against the alternative $\mu/\sigma=P_1$ which is similar with respect to the unknown variance σ^2 of the assumed normal distribution. Unfortunately, the derivation is not complete and is somewhat clumsy. Besides the author seems unaware that his test is the standard t test in slightly disguised form. *H. Chernoff* (Stanford, Calif.).

Noether, Gottfried E. On a theorem of Pitman. Ann. Math. Statist. 26, 64–68 (1955).

The Pitman definition of the asymptotic relative efficiency of two tests is discussed and a theorem of Pitman is extended. A condition is given under which we can use the limit of the ratio of the variances of two test statistics as a measure of asymptotic relative efficiency. Another definition based on the slope of the power function at θ_0 is shown to be equivalent under certain conditions. A third definition is suggested by the author. *M. Sobel* (Allentown, Pa.).

Hoeffding, Wassily, and Rosenblatt, Joan Raup. The efficiency of tests. Ann. Math. Statist. 26, 52–63 (1955).

Given a family $\mathfrak{T}$ of nonsequential tests for testing a hypothesis $\theta\leqq\theta_1$ against $\theta\geqq\theta_2$ let $N(\mathfrak{T})=N(\mathfrak{T};\alpha_1,\alpha_2)$ denote the smallest sample size of any test in $\mathfrak{T}$ whose probabilities of the two kinds of error do not exceed the fixed values α_1, α_2. Under various assumptions the authors derive asymptotic expressions for $N(\mathfrak{T})$ as $\delta=\theta_2-\theta_1\rightarrow0$. The usual Pitman method of fixing θ_1 and letting $\theta_2=\theta_1+kn^{-1/2}$ depend on n is avoided. The relative efficiency of family $\mathfrak{T}_2$ with respect to $\mathfrak{T}_1$ is defined as $N(\mathfrak{T}_1)/N(\mathfrak{T}_2)$ which in certain cases is shown to be independent of α_1 and α_2; this is an extension of the Pitman definition. Illustrative examples are given.
M. Sobel (Allentown, Pa.).

Barnard, G. A. Sampling inspection and statistical decisions. J. Roy. Statist. Soc. Ser. B. 16, 151–165; discussion 166–174 (1954).

"The most important result is . . . that, not only must we know the prior distribution in order to solve a decision problem, but we may have to know it in considerable detail." (From the author's summary.) [To illustrate this, the author points out that the Bayes strategies with respect to two a priori distributions which have some similarity may be quite different (e.g., in the number of observations required for procedures of fixed sample size); however, the important question is not how much the strategies differ, but how much their risk functions differ, and this question is never treated by the author.] *J. Kiefer.*

Bahadur, R. R. A characterization of sufficiency. Ann. Math. Statist. 26, 286–293 (1955).

Denote by X, P, D sample, distribution, and decision space; by x, p, d their elements; by μ, $\cdots$ randomized decision functions; by $L=L_p(d)$ a loss function, and by $r_p(\mu, L)$ the risk function corresponding to μ and L; let $\mathfrak{D}$ be the set of all μ's, and $\mathfrak{D}_T$ the set of decision functions based on a certain statistic $y=T(x)$. $\mathfrak{D}_T$ is said to be essentially com-

plete (L) if to any $\mu \in \mathfrak{D}$ there is a $\nu \in \mathfrak{D}_T$ such that $r_p(\nu, L) \in r_p(\mu, L)$ for all p. It is known that sufficiency of $T(x)$ implies essential completeness of $\mathfrak{D}_T$. The main result of this paper is that the converse is true under a certain condition, the essence of which is that the "response" decision functions corresponding to two different distributions p and q must never be identical. Moreover, it is shown that, under mild restrictions, uniform essential completeness (i.e., completeness for all L) of $\mathfrak{D}_T$ implies sufficiency of $T(x)$. *G. Elfving* (Helsinki).

Miyasawa, Kôichi. On the minimax point estimations. Bull. Math. Statist. 5, no. 3–4, 1–17 (1953).

The first part of the paper develops some theorems on the determinateness of infinite games. Let $X = X_i$ $(i=1, 2, \cdots)$ be a sequence of random variables, F the distribution function of X and Ω the given class of all permissible F; let D be the space of decisions d and $r(F, d)$ the risk function. It is shown that if r is bounded and is a Borel-measurable function of d for every F, and D is compact then the decision problem is determined (Borel sets and compactness are defined through the Wald metric). If, furthermore, $r(F, d)$ is, for given F, a convex function of d and D is also convex then there exists a non-randomized minimax decision rule. The author's methods follow those of Wald [Statistical decision functions, Wiley, New York, 1950; MR 12, 193] and Karlin [Contributions to the theory of games, Princeton, 1950, pp. 133–154; MR 12, 844]. The second part considers special problems. We mention the following result: Consider the family of random variables with mean θ and variance σ^2 satisfying $|\theta| \leq \alpha$ and $L^2 \leq \sigma^2 \leq K^2$. Let $x_1, \cdots, x_n$ be independent observations and confine $\hat{\theta}$, the estimator of θ, to be a linear function of the observations. Then, if the loss function is $(\hat{\theta}-\theta)^2$ the minimax estimator is $\bar{x}/(1+K^2/n\alpha^2)$; if the loss function is $(\hat{\theta}-\theta)^2/\sigma^2$ the minimax estimator is $\bar{x}/(1+L^2/n\alpha^2)$. *A. Dvoretzky* (New York, N. Y.).

Rao, K. S. Testing for serial correlation in a stationary multidimensional discrete stochastic process. Bull. Inst. Internat. Statist. 24, 2ème livraison, 185–194 (1954).

The author considers a general "reduced form" equation in a simultaneous equation model with k endogenous variables and n time periods:

$$x_{st} = E(x_{st}) + \epsilon_{st}; \quad 1 \leq s \leq k, \quad 1 \leq t \leq n,$$

where $E(x_{st})$ is a linear function of lagged values of the k variables. Presumably $E(x_{st})$ could also be a function of certain exogenous variables, but the author does not so state. The hypothesis tested in the paper is that the n normal vectors, $[\epsilon_t]$, are independent against a wide class of alternatives, where

$$\epsilon_t' = [\epsilon_{1t}, \epsilon_{2t}, \cdots, \epsilon_{st}, \cdots, \epsilon_{kt}].$$

Let $E(\epsilon_{s_1 t_1} \epsilon_{s_2 t_2}) = P_{s_1 s_2 \cdot t}$, where $t = |t_1 - t_2|$. The specific hypothesis considered is

$$H_0\colon P_{s_1 s_2 \cdot t} = 0; \quad H_a\colon P_{s_1 s_2 \cdot t} \neq 0 \qquad (t=1, 2, \cdots, n-1; 1 \leq s_1 \leq k).$$

It is possible to transform the ϵ-vectors to mutually independent ξ-vectors. The above test is then equivalent to testing that the dispersion matrices of the ξ-vectors are identical. This is accomplished in a sequential testing procedure, involving successive F-tests.

Unfortunately many printing errors mar the presentation in this article. Also the author fails to indicate explicitly how estimates of the dispersion matrices are obtained from sample data. *R. L. Anderson* (Raleigh, N. C.).

Jowett, G. H. Sampling properties of local statistics in stationary stochastic series. Biometrika 42, 160–169 (1955).

A general theorem is proved which as a typical instance implies that if

$$U = (n-1)^{-1} \sum_1^{n-1} (x_t - x_{t-1})(y_t - y_{t-1}),$$

a statistic which involves "local" comparisons like $x_t - x_{t-1}$ in the parent series x_t, y_t $(t=1, \cdots, n)$, the variance of U depends essentially on the second central differences $\Delta_s''(x)$, $\Delta_s''(y)$ of the autocorrelations $\rho_s(x)$, $\rho_s(y)$ of x_t, y_t, the point being that the Δ_s'' tend rapidly to zero provided the ρ_s are approximately linear over s-intervals of width twice the time unit. Thus, broadly speaking, the variational properties of "local" statistics depend primarily on local variational properties of the parent series. Further examples and hints on industrial applications are given. *H. Wold* (Uppsala).

Lampard, D. G. A new method of determining correlation functions of stationary time series. Proc. Inst. Elec. Engrs. C. 102, 35–41 (1955).

The author considers the linear expansion of the correlation function of a time series in orthogonal functions. He shows that the coefficients in this expansion may be evaluated by feeding the time series as an electric signal into a suitable filter, and measuring the time average of the product of filtered and original signals. Details are given of an apparatus which will evaluate the sum of the first 10 terms of a Laguerre polynomial expansion. *P. Whittle.*

Cruickshank, A. J. O. A note on time series and the use of jump functions in approximate analysis. Proc. Inst. Elec. Engrs. C. 102, 81–87 (1955).

The author approximates the input and output of a filter by step-functions with steps of fixed length, and terms the quotient of the Laplace transforms of these two step-functions the "jump-transfer function" of the filter. The relation between the ordinates of the step-functions is found by methods analogous to and derivable from those of the conventional operational calculus. *P. Whittle.*

Castellano, Vittorio. Contributo allo studio delle serie cicliche. Statistica, Bologna 15, 23–56 (1955).

Yanoši [Janossy], L. Statistical problems of an electron multiplier. Ž. Eksper. Teoret. Fiz. 28, 679–694 (1955). (Russian)

The first part of this article is a somewhat modified Russian version of the author's earlier papers [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 1, 357–367 (1951); Acta Math. Acad. Sci. Hungar. 2, 165–176 (1951); MR 13, 957; 14, 388]. The second part is an extension of the previous work. The author determines here the probability distribution of the secondary electrons produced by a primary electron from the distribution of the electrons emitted by the Nth electrode. *E. Lukacs* (Washington, D. C.).

Olsen, H., Wergeland, H., and Øverås, H. On a statistical problem in emulsion microscopy. Norske Vid. Selsk. Forh., Trondheim 28, 25–29 (1955).

Let $\{y_1(x), 0 \leq x < \infty\}$ be a Brownian motion process, and let $y(x) = \int_0^x y_1(s)ds$. The author finds the conditional distribution of $y(x)$, $0<x<L$, under the hypothesis that $y(L)=0$, and with $y_1(0)$ either fixed or uniformly distributed

on $[-\frac{1}{2}, \frac{1}{2}\pi]$. In the latter case the conditional distribution is normal, with mean 0 and variance proportional to $x^2(1-x/L)^2$. *J. L. Doob* (Urbana, Ill.).

Theory of Games, Mathematical Economics

★Blackwell, David, and Girshick, M. A. Theory of games and statistical decisions. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1954. xi+355 pp. $7.50.

The book is intended primarily as a textbook for first-year graduate students. It restricts the formal development to discrete distributions and continuous ones are used only for the purpose of illustration. This and other restrictions, while limiting the generality of the theory, make it possible to by-pass completely the delicate measure-theoretic and functional-analysis considerations so characteristic of most advanced work in decision theory. This book is the first comprehensive elementary introduction to statistics in the modern spirit, and a great part of the material covered has never before appeared in book form. It is carefully written and, except for a gap in the proof of Theorem 10.9.1, seems to be rather free from lapses. Its style is formal and this may render it unsuitable for self-study; on the other hand the many problems should prove of great help to both student and instructor. There is a rather complete bibliography but no attempt is made to discuss the historical development of the theory. The choice and ordering of the material naturally reflect to some degree the personal tastes and interests of the authors.

The following abbreviated list of contents should give some idea of the book's scope. 1. Games in normal form (p. 1). 2. Values and optimal strategies in games (p. 30). 3. General structure of statistical games (p. 75). 4. Utility and principles of choice (p. 102). 5. Classes of optimal strategies (p. 121). 6. Fixed-sample-size games with finite Ω (p. 133). 7. Fixed-sample-size games with finite A (p. 171). 8. Sufficient statistics and the invariant principle in statistical games (p. 208). Sequential games (p. 237). 10. Bayes and minimax sequential procedures when both Ω and A are finite (p. 261). References (p. 337). Index (pp. 347–355). *A. Dvoretzky* (New York, N. Y.).

Karlin, Samuel. The theory of infinite games. Ann. of Math. (2) **58**, 371–401 (1953).

The author develops a general abstract theory of games, replacing the probability distributions and the kernel by suitable Banach spaces and operators, respectively. The very generality of the approach and comprehensiveness of the definitions make impossible a concise and precise reproduction of specific results. The determinateness of a game is shown to be implied by rather weak conditions and it is proved that every game can be extended to a determinate one; the Wald theory of completeness, admissibility and Bayes solutions is also carried over to the present general set-up. Next, a perturbation theory is developed studying the effect of variations in the operators and sets of strategies on the value of a game; this value is shown to be a continuous functional, in an appropriate topology. The application of the general results is illustrated and certain types of games, among them those with constraints, are considered in some detail. The paper also contains pertinent remarks on the relation between the theory of games and that of groups of transformations. Through most of the paper only linear and bounded operators are considered, but it is indicated how to construct a non-linear theory.
A. Dvoretzky (New York, N. Y.).

Mycielski, Jan, and Zięba, A. On infinite games. Bull. Acad. Polon. Sci. Cl. III. **3**, 133–136 (1955).

The authors consider some infinite games in extensive form with perfect information at each move. Their main results have already been obtained by D. Gale and F. M. Stewart [Contributions to the theory of games, v. II, Princeton, 1953, pp. 245–266; MR **14**, 999]. The references are chiefly to work in game theory by Polish mathematicians.
W. H. Fleming (Lafayette, Ind.).

Danskin, J. M. Fictitious play for continuous games. Naval Res. Logist. Quart. **1** (1954), 313–320 (1955).

By a continuous game is meant a game over compact spaces X and Y with payoff $M(x, y)$ continuous on $X\times Y$. The convergence of the analogue for continuous games of the Brown-Robinson iterative process [G. W. Brown, Activity analysis of production and allocation, Wiley, New York, 1951, pp. 374–376; MR **15**, 48; J. Robinson, Ann. of Math. (2) **54**, 296–301 (1951); MR **13**, 261] is established. The proof is modelled on Mrs. Robinson's.
W. H. Fleming (Lafayette, Ind.).

Bellman, Richard. Decision making in the face of uncertainty. II. Naval Res. Logist. Quart. **1** (1954), 327–332 (1955).

[For part I see same Quart. **1**, 230–232 (1955); MR **16**, 730.] This paper considers multi-stage processes involving both zero-sum and non-zero sum games. Using the concept of "games of survival" approximate solutions are derived for both classes of multi-stage games under various realistic assumptions. (Author's summary.) *L. J. Savage.*

Fréchet, Maurice. Sur l'importance en économétrie de la distinction entre les probabilités rationnelles et irrationnelles. Econometrica **23**, 303–306 (1955).

Stresses the difference between actual economic behavior and behavior of that idealized man to whom the formal theory of personal (that is, subjective) probability applies.
L. J. Savage (Chicago, Ill.).

Prager, William. On the role of congestion in transportation problems. Z. Angew. Math. Mech. **35**, 264–268 (1955). (German, French and Russian summaries)

In discussing the most efficient manner of distributing a product from several sources to numerous locations, Hitchcock assumed that the specific transportation cost for a given shipping route was independent of the amount shipped over this route. In the present paper the problem is discussed under the assumption that the specific shipping cost for a route increases linearly with the amount shipped over this route. It is shown that the optimal shipping program then is unique and that it can be discussed either in terms of flows along the routes or in terms of accounting prices at the centers of production and consumption. A maximum principle is established for the accounting prices, which leads to a maximum characterization of the cost of the optimal program. *T. L. Saaty* (Washington, D. C.).

Kneser, Hellmuth. Soziologie und Wirtschaftswissenschaft in heutiger mathematischer Behandlung. Studium Gen. **6**, 666–678 (1953).

von Mises, Ludwig. Bemerkungen über die mathematische Behandlung nationalökonomischer Probleme. Studium Gen. 6, 662–665 (1953).

Mathematical Biology

*Bush, Robert R., and Mosteller, Frederick. **Stochastic models for learning.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1955. xvi+365 pp. $9.00.

This important book is concerned with the formulation and application of a general model for the results of certain learning experiments. In Part I (Chapters 1–8) an abstract theory is developed, the elements of which may be identified in various ways with experimental features. The theory relates to a series of "trials" at each of which is presented a set of r mutually exclusive and exhaustive "responses" (e.g. turning in different directions in a maze), the probabilities for which are transformed at each trial by a linear operator depending on one of t "events". Various restrictions on the parameters arise naturally since the probabilities are bounded between 0 and 1. (The use of linear operators is supported in Chapter 2 by a model of stimulus sampling and conditioning, although identification with a particular psychological theory is unnecessary.) Most of the detailed development concerns the case $r=2$. Suppose p is the probability of response A_1, and let event E_i be associated with operator Q_i, where $Q_i p=\alpha_i p+(1-\alpha_i)\lambda_i$; λ_i is called the "limit point" of the operator. The general problem is to investigate the change of p from trial to trial. Three types of experiment are considered:

I. "Experimenter-controlled events", the order of which is independent of the responses. For (i) a fixed order, the solution is given for repetitions of one operator, and (for $r=t=2$) for operators occurring in repeated cycles. Conditions for the commutativity of two operators are given. Repeated operation of a single operator with $\lambda=0$ leads to problems in run theory, and tables are given for use in calculating the first two moments of run length. For (ii) two events (e.g. reward and non-reward) occurring at random with fixed probabilities, the mean value of p at the nth trial, and a simple difference equation for the moments, are obtained. For (iii) two events forming a Markov chain, and $\alpha_1=\alpha_2$, a recurrence formula is obtained for the mean.

II. "Subject-controlled events", where event E_i is an occurrence at the previous trial of response A_i. The difference equation for the moments is, in general, inconvenient but simplifies for equal α_i, when the mean and variance are obtained explicitly for general t. For unequal α_i, and $t=2$, approximate results for the mean are obtained by Monte Carlo methods, and bounds are derived analytically. Results are given relating to the asymptotic distribution of p. Many results simplify when one of the λ_i is 0 and the other is 1, or when the operators commute. In Section 4.7 some important theorems about the distributions of p are stated without proof.

III. "Experimenter-Subject controlled events", where E_i is the combination of response A_j and "outcome" O_k, where O_k (e.g. reward or non-reward) is chosen with probability π_{jk} when A_j occurs. A recurrence formula for the moments is obtained for $r=2$.

Part II (Chapters 9–15) starts with a general chapter on identification, estimation, and goodness-of-fit, and then describes applications of the theory to experiments on free-recall verbal learning, avoidance training, imitation, problems with symmetric choice, and runway experiments. These experiments involve both animal and human subjects. For some problems maximum likelihood methods are given for estimation of parameters, but more often less efficient methods are proposed. The model is remarkably flexible, and appears to the reviewer to provide impressive fits to the data. It is ingeniously adapted to describe distributions of latency period in runway experiments, by supposing that each period is the sum of a random number of random variates, the parameter of the process at trial n depending on a probability p_n following the linear-operator model.

The book, which is admirably written, is intended primarily for the experimental psychologist, and advanced mathematics is avoided. It nevertheless provides the mathematician with a useful summary of a subject which, in spite of its successful start, is not lacking in unsolved problems.

P. Armitage (London).

TOPOLOGY

*Kelley, John L. **General topology.** D. Van Nostrand Company, Inc., Toronto-New York-London, 1955. xiv +298 pp. $8.75.

This text-book is intended to provide a background for "modern analysis" (which seems to mean the theory of linear spaces). It is roughly similar in scope to the first volume of Bourbaki's "Topologie générale" [2nd ed., Actualités Sci. Ind., no. 1142, Hermann, Paris, 1951; for a review of the 1st ed. see MR 3, 55], though with variations of emphasis. The principal difference is that convergence is dealt with by "nets" (directed systems) instead of filters; the method lends itself to intuitively satisfactory applications. There are up-to-date discussions of paracompactness (including some new material) and metrizability (including the Nagata-Smirnov theorem). Other features include a chapter on function spaces, and a remarkable appendix which develops set-theory from the axioms down to ordinal and cardinal numbers, in a manner which could easily be "formalised" in the sense of mathematical logic. Throughout the book, the treatment reveals modern refinements, which are shown partly in the elegance of the methods, and partly (perhaps less significantly) in the weakening of assumptions. For example, the Lindelöf property is used, where possible, instead of the assumption of a countable base, and "point" separation axioms are to some extent avoided, pseudo-metric spaces playing an important role. Occasionally this avoidance (particularly of the T_1 axiom) leads in fact to more being assumed than is quite necessary; but the author has wisely avoided any systematic attempt to reduce all assumptions to a minimum. As with Bourbaki, a valuable feature of the book is the collection of exercises; besides routine exercises, these include further (often extensive) theoretical developments and applications (with suitable hints), and some useful counterexamples. There is very little on the geometrical aspects of the subject; local connectedness receives little more than mention, and arcs are not mentioned at all.

A more detailed account of the contents now follows. Chapter 0, Preliminaries, rapidly derives the fundamental properties of sets, relations, functions (identified with their

graphs), orderings, groups, rings, fields, linear spaces, real numbers (introduced via ordered fields), integers, definition by induction, decimals, countable sets, ordinal and cardinal numbers, cartesian products, and forms of the axiom of choice (including those of Hausdorff and Zorn). A firm foundation for the logical ideas involved, and the definition and basic properties of the ordinal numbers, are supplied in the appendix.

Chapter 1, Topological spaces, begins with the definition (unmotivated) of a "topological space", the "topology" being the family of open sets (other formulations are indicated in the exercises at the end of the chapter); then follow neighborhood systems, closed sets, accumulation (=limit=cluster) points, closure, interior, boundary, bases, sub-bases, and connectedness. In comparing topologies, the author replaces the terms "finer" and "coarser" (which are themselves replacements for the older "weaker" and "stronger", in some order) by "larger" and "smaller", which however are less appropriate in other approaches to the subject. The illustrative examples are few but significant.

Chapter 2 is on Moore-Smith convergence. The author's elegant exposition of the Moore-Smith-Birkhoff-Tukey theory follows his paper [Duke Math. J. 17, 277–283 (1950); MR 12, 194]; the exercises include applications to integration, the parallel theory of filters, and "universal" nets (corresponding to ultrafilters).

Chapter 3, Product and quotient spaces, deals with the elementary properties of continuous functions and homeomorphisms, and goes on to consider Cartesian product spaces, and quotient spaces and the associated "projections", including the relation between closed mappings and upper semi-continuous decompositions, but not the general notion of a "quasicompact" ("stark-stetig") mapping. Here and elsewhere in dealing with product spaces, the author makes a very common oversight; desirable properties of the factor spaces cannot as a rule be inferred from those of the product—if one factor is empty. The exercises include applications to topological groups and factor groups, and real linear spaces and linear functionals. [The reviewer believes that the example in Exercise R, p. 104, is wrongly attributed; it was given by J. H. Roberts, Amer. J. Math. 52, 551–562 (1930).]

Chapter 4, Embedding and metrization, begins with normality, regularity, and Urysohn's lemma; Tychonoff's embedding theorem for completely regular T_1 spaces is deduced from a versatile lemma on "enough continuous functions". Metric spaces are now introduced for the first time; they are handled as special cases of pseudo-metric spaces. Urysohn's metrization theorem (for separable spaces) follows quickly; that of Smirnov and Nagata (the general case) concludes the chapter. The exercises include further properties of metric spaces, and more about separation axioms (theorems and counterexamples); perfect normality is included, but not complete normality. [Tychonoff's regular but not completely regular space, or a reference to it, might have been included here.]

Chapter 5, Compact spaces, concerns spaces for which every open "cover" (=covering) has a finite sub-cover; no separation axioms are assumed. The usual equivalent statements about closed sets and nets are derived, and conditions under which sequences suffice are considered. (Countable and sequential compactness, and complete limit points, appear in the exercises.) Separation axioms in compact spaces are discussed; here some results of Yang [Proc. Amer. Math. Soc. 5, 185–189 (1954); MR 15, 976] might have mitigated the awkwardness caused by the desire to avoid the T_2 axiom. Tychonoff's product theorem is given two proofs, one depending on Alexander's sub-base theorem; it would have been more consistent (and shorter) to have replaced the other proof by one using universal nets. After brief sections on locally compact spaces and quotient spaces under closed mappings, the Alexandroff and Stone-Cech-Tychonoff compactifications are considered in some detail. (Wallman's construction is sketched in an exercise.) A general form of Lebesgue's covering lemma leads up to a discussion of paracompactness, which includes recent results of Michael, Griffin and the author. As one would expect, this chapter has more exercises than any other. [It should be remarked that, in exercise U, p. 170, the meaning given to "star-refinement" is not Tukey's original meaning, but what Tukey calls "Δ-refinement"; however, they are equivalent for the purpose in hand.]

Chapter 6, Uniform spaces, defines a uniformity on X in terms of a family of sets containing the diagonal of $X \times X$; some alternative definitions are listed in the exercises. Then some properties of the uniform topology, uniform continuity, subspaces and products of uniform spaces. Pseudo-metrizable uniformities are characterized, and for any uniform space X the relation between the uniformity and the "gage" (the family of uniformly continuous pseudo-metrics) is discussed. Completeness is defined and handled by means of "Cauchy nets"; these make it obvious that a product of complete spaces is complete, and give a perspicuous construction of the completion of X. Compact and totally bounded (=precompact) uniformities are considered; and the chapter ends with properties of complete pseudo-metric spaces, including Baire's theorem (but with the traditional term "first category" replaced by "meager"). The exercises naturally include "absolute G_δ's" and applications to topological groups (to which the closed-graph theorem is generalized). [The term "topologically complete", used in one of the exercises, has unfortunately been used with a different meaning by Čech, Ann. of Math. (2) 38, 823–844 (1937).]

Finally Chapter 7, Function spaces, applies the general theory to the discussion and comparison of various topologies (pointwise, compact-open, uniform, uniform on compact sets) on sets of mappings from X to Y. The requirement of "joint continuity" (that $f(x)$ depend continuously on the pair f, x) is investigated, and criteria are obtained for the compactness of sets of equicontinuous or "evenly continuous" (roughly, locally equicontinuous) functions. The exercises include the Stone-Weierstrass approximation theorem (suitably led up to) and results on almost periodic functions on groups (the relation to Bohr's original definition being left a little obscure; one of the properties mentioned coincides with one of Bohr's theorems, though not with his definition).

The appendix on set-theory has already been mentioned; it travels rapidly over a great deal of ground, and virtually demands some acquaintance with mathematical logic, but should be valuable for reference. There is a quite extensive bibliography, and an index.

The author has been more concerned to expound the main ideas and methods, and to illustrate the wide range of applications, than to treat any topic very thoroughly. His style also places emphasis on essentials at the expense of details. There are many illuminating comments, in a conversational style, which make the arguments easy to follow in the large. However, the speed with which the foundations are laid, and the not infrequent omission of small steps in

the reasoning and of "obvious" routine theorems, may sometimes puzzle a beginner. (For example, he is left to discover for himself, unprompted, that the union of two compact sets is compact—a result proved later—and that each member of a uniformly continuous family of functions is continuous.) There are several misprints in the cross-references, particularly to the exercises; the exercise meant is then usually a neighbor of the one referred to. There are other obvious misprints on p. 131, line 24; p. 146, line 20; p. 204, lines 16, 17; p. 220, line 6. *A. H. Stone.*

Iséki, Kiyoshi. On hypocompact spaces. Portugal. Math. 13, 149–152 (1954).

A Hausdorff space is called hypocompact if every open covering has an open star-finite refinement. Concerning such spaces, the author proves results which are analogous to some results on paracompact spaces obtained by the reviewer [Proc. Amer. Math. Soc. 4, 831–838 (1953); MR 15, 144]. In particular, it is shown that every F_σ-subset of a hypocompact space is hypocompact, and that consequently the cartesian product of a hypocompact space and a σ-compact space is hypocompact. *E. Michael.*

Blair, Robert L. Stone's topology for a binary relation. Duke Math. J. 22, 271–280 (1955).

Let L be a set and R a binary relation defined on L. For $M\subseteq L$, let M^- (M^+) denote the set of all $x\in L$ such that xRy (yRx) for every $y\in M$. Let S be a fixed subset of L and M a subset of S. The closure $e(M)$ of M in S is defined by $e(M)=S\cap M^{-+}$. Then the mapping $M\rightarrow e(M)$ is a closure operation. We shall say that S admits the Stone topology (relative to R) if the closure of the empty set is empty and if $e(M)\cup e(N)\supseteq e(M\cup N)$. An element $x\in L$ is said to be s-irreducible (relative to R) in case (i) $yR'x$ for some $y\in L$ (R' is the complement of R) and (ii) for any two-element subset $\{a,b\}$ of L, $\{a,b\}^-\subseteq\{x\}^-$ implies that aRx or bRx. A lattice L is called an F-lattice in case each element of L is a meet of meet-irreducible elements. A subset X of a topological space S is called d-dense in S in case $F\cap X$ is dense in F for every closed set F of S.

Some results: If R is a transitive binary relation defined on a set L and if S is a set of s-irreducible elements of L, then S admits the Stone topology. A lattice L is isomorphic to the lattice of all open sets of some T_0-space if and only if L is a complete distributive F-lattice. Let L be a complete distributive F-lattice. Then a T_0-space X has an open-set lattice isomorphic to L if and only if X is homeomorphic to a d-dense subset of $S(L)$, where $S(L)$ is the set of all s-irreducible elements of L topologized by the Stone topology. *M. Novotný* (Brno).

Michael, Ernest. Point-finite and locally finite coverings. Canad. J. Math. 7, 275–279 (1955).

The author proves that every point-finite (open) covering of a collectionwise normal space has a locally finite refinement. He shows by examples that, even for perfectly normal spaces, the requirement of collectionwise normality cannot be omitted, but is not always necessary for the truth of the theorem. *A. H. Stone* (Manchester).

Nagami, Keiô. Paracompactness and strong screenability. Nagoya Math. J. 8, 83–88 (1955).

The author's principal results can be summarized as follows: (1) A regular space is paracompact if and only if it is strongly screenable in the sense of Bing [Canad. J. Math. 3, 175–186 (1951); MR 13, 264]. (2) Every point-finite covering of a collectionwise normal space [Bing, ibid.] has a locally finite refinement. [Reviewer's note: (1), as the author remarks in an addendum, was obtained by the reviewer in Proc. Amer. Math. Soc. 4, 831–838 (1953) [MR 15, 144]; (2) was obtained independently by the reviewer in [the paper reviewed above]. The paper concludes by showing that every normal, countably paracompact screenable space [Bing, ibid.] is strongly screenable. *E. Michael* (Seattle, Wash.).

Pierce, R. S. Coverings of a topological space. Trans. Amer. Math. Soc. 77, 281–298 (1954).

Let P be a semi-lattice (a partially ordered set with a zero element in which every finite subset has an inf). Let S_P be the space of all proper dual ideals of P, with the sets $\{X\in S_P|a\in X\}$, $a\in P$, taken as an open basis. Subspaces of S_P are called P-spaces. The set of all compact T_1-P-spaces is denoted Σ_P. A non-empty collection $\mathfrak{a}$ of finite sets $\alpha\subset P$, $\alpha\neq 0$, is called a P-covering ideal if (1) $\alpha_1,\alpha_2\in\mathfrak{a}$ implies $\alpha_1\wedge\alpha_2\in\mathfrak{a}$ (where $\alpha_1\wedge\alpha_2$ is the set of all $a_1\wedge a_2$, $a_i\in\alpha_i$, (2) $\alpha\in\mathfrak{a}$, $\beta>\alpha$ implies $\beta\in\mathfrak{a}$ (where $\beta>\alpha$ means that, for any $a\in\alpha$, $b\geqq a$ for some $b\in\beta$). The collection of all P-covering ideals is denoted by Γ_P.

Main result: there is a Galois connection [cf. G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., vol. 25, rev. ed., New York, 1948; MR 10, 673] between Σ_P and Γ_P defined by the mappings $S\rightarrow\mathfrak{a}(S)$, $\mathfrak{a}\rightarrow S(\mathfrak{a})$, where $\mathfrak{a}(S)$ is the set of all finite $\alpha\in P$, $\alpha\neq 0$, such that $X\cap\alpha\neq 0$ for all $X\in S$, $S(\mathfrak{a})$ consists of all $J\in S_P$ such that (1) $\alpha\cap J$ for all $\alpha\in\mathfrak{a}$, (2) J is minimal with regard to property (1).

There are many other results concerning subspaces and continuous mappings of P-spaces. Some results, mostly known, concerning rings, lattices and topological spaces are deduced from the main theorem; e.g. well known compactifications of T_1-space R are obtained as $S(\mathfrak{a})$, where $\mathfrak{a}$ is the set of all finite (finite normal) open coverings of S (P is the collection of all open subsets of R). *M. Katětov.*

Mikulík, Miloslav. Metric lattices. Czechoslovak Math. J. 4(79), 364–371 (1954). (Russian. English summary)

Let S be a complete lattice which is a compact metric space. Suppose that the diameter of every non-void subset $A\subset S$ equals the distance between sup A and inf A. In such a lattice metric convergence, order-convergence and star-convergence are identical. A non-void convex sublattice $A\subset S$ is closed [see G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., vol. 25, rev. ed., New York, 1948, p. 50; MR 10, 673] if, and only if A is compact. The author's concept of metric lattice differs from that of Wilcox and Smiley [Ann. of Math. (2) 40, 309–327 (1939)] and of Birkhoff [loc. cit., p. 76]. The metric lattice studied by the author is realized by a suitable system of solutions of the differential equation $x'=f(t,x)$. *M. Novotný* (Brno).

Roberts, G. T. Topologies defined by bounded sets. Proc. Cambridge Philos. Soc. 51, 379–381 (1955).

The author announces some results concerning the notion of boundedness in linear spaces. Let X be a linear space with a distinguished family $\mathfrak{B}$ of subsets, called bounded sets, and let X' be the space of all linear functionals which are bounded on each member of $\mathfrak{B}$. Under rather weak assumptions on $\mathfrak{B}$, the following descriptions yield the same topology for X: (a) the strongest locally convex topology making members of X' continuous; (b) the topology which has as a base for the neighborhoods of 0 the family of all convex circled sets which absorb each member of $\mathfrak{B}$; (c) the

topology of uniform convergence on subsets of X' which are bounded relative to the topology of uniform convergence on members of $\mathfrak{B}$; and (d) the same as (c), with $\mathfrak{B}$ replaced by the family of all weakly-(X, X') bounded subsets of X. The space X, with the topology thus described, is a tonnelé space if certain sequential completeness conditions hold.

J. L. Kelley (Berkeley, Calif.).

Hayashi, Yoshiaki. On the dimension of topological spaces. Math. Japon. 3, 71–84 (1954); errata, 136 (1955).

Let $H_0(R)$ be the family of all closed sets in R (an arbitrary topological space). Let $H_n(R)$ be the family of all sets C such that either $C=A\cup B$ or $C=A-B$ with A and B in $H_{n-1}(R)$. Let $H(R)=\bigcup_{n=0}^{\infty} H_n(R)$. If E is an element of $H(R)$, it is called a quasi-closed set of R. The author defines a dimension of the space R as follows: 1) If R is empty, $\dim R=-1$. 2) If R is the union of a countable collection $\{R_i\}$ of quasi-closed sets of R with the Urysohn-Menger dimension of each $R_i\leqq 0$, then $\dim R\leqq 0$. 3) If $R=R_1\cup R_2$ with $\dim R_1\leqq n-1$ and $\dim R_2\leqq 0$, then $\dim R\leqq n$. 4) $\dim R=\inf\{n: \dim R\leqq n\}$.

The author shows that $R'\subset R$ implies $\dim R'\leqq \dim R$ (the monotone property), that the countable union of quasi-closed sets of dimension $\leqq n$ has dimension $\leqq n$ (this implies the sum theorem), and that $\dim R\leqq n$ if and only if there exist $(R_1, R_2, \ldots, R_{n+1})$ with $R=R_1\cup R_2\cup\ldots\cup R_{n+1}$ and $\dim R_i\leqq 0$ (the decomposition theorem). He also proves that for separable metric spaces his dimension coincides with the Urysohn-Menger and Lebesgue dimension.

Haskell Cohen (Baton Rouge, La.).

de Groot, J., and de Vries, H. A note on non-Archimedean metrizations. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17, 222–224 (1955).

A "non-Archimedean" metric is one which satisfies the triangle axiom in the strong form

$$\rho(x, z)\leqq\max\{\rho(x, y), \rho(y, z)\}.$$

Spaces which can be given such metrics have been characterized by de Groot; they coincide with 0-dimensional metric spaces (dimension being defined in terms of neighborhoods of closed sets). The authors show that a paracompact Hausdorff space can be given a non-Archimedean metric if every point has a neighborhood which can be given such a metric. They obtain several corollaries; e.g., if X is a metric space in which each point has a neighborhood with less than c points, then X has a non-Archimedean metric. Their main results are contained in the sharper theorem [Morita, Math. Ann. 128, 350–362 (1954); MR 16, 501] that 0-dimensional metric spaces form a normal family. [In the statement of property α, p. 222, it should evidently be required that H is metrizable.]

A. H. Stone (Manchester).

Kuratowski, K. Un théorème sur les espaces complets et ses applications à l'étude de la connexité locale. Bull. Acad. Polon. Sci. Cl. III. 3, 75–80 (1955).

Let X be a metric space and $\{f_n\}$ a sequence of continuous real functions defined on X such that every sequence $\{x_n\}$ of points of X satisfying the following two conditions is convergent: (1) The Cauchy condition; (2) for every m, the sequence $\{f_m(x_n)\}$ is convergent. Then it is shown how to construct a new distance function for X which renders X a complete space. The well-known theorem of Alexandroff ("every G_δ subset of a complete space is homeomorphic to a complete space") is an immediate corollary. Also, the theorem to the effect that the space of lc^n closed subsets of a compactum is metric (P. A. White) and complete (E. G. Begle) is a corollary, as is also the corresponding theorem where "LC^n" replaces "lc^n".

R. L. Wilder.

Blumenthal, Leonard M. An extension of a theorem of Jordan and von Neumann. Pacific J. Math. 5, 161–167 (1955).

Let M be the class of complete, metrically convex, and externally convex metric spaces. [For definitions see Blumenthal, Theory and applications of distance geometry, Oxford, 1953; MR 14, 1009.] Say that a metric space E has the ith euclidean four-point property, ie4pp, where $i=0$ (unqualified), 1 (weak) or 2 (feeble), when every quadruple p, q, r, s of points of E which satisfies the conditions (C_j), $j\leqq i$, is isometrically embeddable in a euclidean space: (C_1) $pq+qr=pr$. (C_2) $pq=qr$. W. A. Wilson [Amer. J. Math. 54, 505–517 (1932)] showed that the 0e4pp characterizes generalized euclidean (ge) spaces, that is, linear spaces with norm determined by an inner product, among the spaces E in M. Blumenthal [op. cit., page 127] showed the same for the 1e4pp. Here he shows that if $E\in M$, then 2e4pp implies 1e4pp, and also that the condition of Jordan and von Neumann [Ann. of Math. (2) 36, 719–723 (1935)], which characterizes ge spaces among normed linear spaces (nls), is equivalent in an nls to the 2e4pp.

An example, a convexly metrized tripod, is given to show that the ptolmaic inequality, which Schoenberg [Proc. Amer. Math. Soc. 3, 961–964 (1952); MR 14, 564] showed characterizes ge spaces among semimetric linear spaces, may hold in an E of M which is not isometrically embeddable in any ge space.

M. M. Day (Urbana, Ill.).

Ishihara, Tadashige. On multiple distributions. Proc. Japan Acad. 30, 352–357 (1954).

The author treats various topologies on spaces of differentiable functions with compact support on euclidean space and on direct products of such spaces, using a modification of work of B. H. Arnold [Duke Math. J. 18, 631–642 (1951); MR 13, 147]. He then states a number of results concerning continuity, uniqueness, etc. of operations on spaces of distributions (in the general sense of L. Schwartz) to provide a basis for a study of multiple distributions.

I. E. Segal (Chicago, Ill.).

Ishihara, Tadashige. Addenda to "On multiple distributions." Osaka Math. J. 7, 129–130 (1955).

See same J. 6, 189–205 (1954); MR 16, 373.

Tumarkin, L. A. On rational one-dimensional compacta. Vestnik Moskov. Univ. 8, no. 8, 73–78 (1953). (Russian)

Following a similar definition by Urysohn, the author defines an n-dimensional compactum R to be rational if there is a countable dense subset N of R with $R-N$ of dimension $<n$. If, for each countable dense N, $R-N$ is of dimension $<n$, then R is said to be homogeneously rational. Suppose R is a one-dimensional compactum. Then R is proved to be homogeneously rational if and only if it contains no continuum of condensation (that is, contains no nowhere-dense continuum). Moreover, R is nonhomogeneously rational if and only if R contains no points of Menger-Urysohn order c and contains a continuum of condensation. Finally, R is irrational if and only if R contains a point of order c. Several problems are also listed.

E. E. Floyd (Charlottesville, Va.).

Homma, Tatsuo, and Kinoshita, Shin'ichi. On homeomorphisms which are regular except for a finite number of points. Osaka Math. J. 7, 29–38 (1955).

A set X is called a C^*-set if $X-A$ is connected for any finite subset A of X. If h is a homeomorphism of X onto X and $p \varepsilon X$, h is said to be regular at p provided h has equicontinuous powers at p. Let X be a compact metric C^*-set and let h be a homeomorphism of X onto itself. The following two theorems are the principal results of the paper. I. If h is regular except at a finite number of points of X, then the number of points at which h is not regular is at most two. II. If h is not regular at the distinct points $a,b \varepsilon X$ but h is regular at all other points of X, then either (1) for each $x \varepsilon X-b$ the positive semi-orbit of x under h converges to a and for each $x \varepsilon X-a$ the negative semi-orbit of x under h converges to b, or (2) for each $x \varepsilon X-a$ the positive semi-orbit of x under h converges to b and for each $x \varepsilon X-b$, the negative semi-orbit of x under h converges to a.

The converse of II, except that X may be taken as any compact metric space, has been shown in an earlier paper by these authors [J. Math. Soc. Japan 5, 365–371 (1953); MR 15, 730]. Another result of the authors [Osaka Math. J. 6, 135–143 (1954); MR 16, 160] combined with II is used to show that when h is a homeomorphism of S^3 onto itself which is regular except at exactly two points, then h is topologically equivalent to a dilation of S^3. *W. R. Utz.*

Kosiński, A. A theorem on monotone mappings. Bull. Acad. Polon. Sci. Cl. III. 3, 69–72 (1955).

It is proved that if $f: X \rightarrow Y$ is a monotone interior mapping of a compact metric space X into a finite-dimensional space Y, with each $f^{-1}(x)$ non-degenerate, then given $\epsilon > 0$ there exists a finite collection $G_1, \dots, G_k$ of open sets with disjoint closures such that diam $G_i < \epsilon$ and f maps $\sum G_i$ onto X. A somewhat stronger theorem has been proved by J. L. Kelley [Trans. Amer. Math. Soc. 52, 22–36 (1942); MR 3, 315]. *E. E. Floyd* (Charlottesville, Va.).

***Whyburn, G. T. Introductory topological analysis.** Lectures on functions of a complex variable, pp. 1–14. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Ces pages sont une introduction générale au Colloque d'Ann Arbor. L'auteur expose clairement les problèmes de la theorie des fonctions analytiques qui sont de nature purement topologique, et montre le succès de l'utilisation de l'indice de circulation dans ce genre de question. Cet indice de circulation permet en particulier de caractériser des applications plan sur plan continues et possédant les propriétés topologiques des applications différentiables. Ainsi sont démontrées par voie exclusivement topologique des propriété locales des transformations intérieures, le théorème de Rouché ainsi que des propriétés globales dans le cas d'une application d'une surface compacte sur une autre; l'auteur passe ensuite à la convergence des suites de transformations intérieures. Cet exposé tout à fait remarquable par l'élégance et la simplicité des méthodes de démonstration qui y sont indiquées, est susceptible d'attirer dans cette voie de nouveaux chercheurs. *L. Fourès* (Princeton, N. J.).

***Baum, Walter. A topological problem originating in the theory of Riemann surfaces.** Lectures on functions of a complex variable, pp. 405–407. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Soient F_p, F_q deux surfaces fermées de genre p et q respectivement; f une application uniforme continue de F_p dans F_q. Soit c le degré de l'application f. L'inégalité de Kneser $p-1 \geqq |c|(q-1)$ est une extension topologique de la formule d'Hurwitz-Riemann. L'inégalité de Kneser a été démontrée par L. Robinson [Thèse, Syracuse Univ., 1953] par voie algébrique et étendue à une application simpliciale d'un complexe C_p sur C_q; p, q, c sont alors définis algébriquement par les groupes fondamentaux et les groupes de Hopf associés à C_p et C_q et la classe d'homotopie de f. Les groupes fondamentaux de C_p et C_q sont en fait isomorphes à ceux des surfaces fermées de genre p et q. *L. Fourès.*

Hukuhara, Masuo. Sur l'application qui fait correspondre à un point un continu bicompact. Proc. Japan Acad. 31, 5–7 (1955).

Motivated by the theory of the initial-value problem for an ordinary differential equation of order 1 (concerning cases where the solution is not necessarily unique), the author proves some results concerning mappings $\mathfrak{F}$ which map a point x of some space R into a set $\mathfrak{F}(x)$ of a space $\mathfrak{R}$. As an example we mention the following one: If for each x in the domain of $\mathfrak{F}$ the set $\mathfrak{F}(x)$ is bicompact and if $\mathfrak{F}$ is "upper semi-continuous", then the image of a bicompact set is bicompact. As an application it is shown that the set of continuous solutions u of the Volterra integral equation (1) of a paper by Satō [same Proc. 31, 1–4 (1955); MR 16, 1120] is a bicompact set in the space (C) defined in that review. *E. H. Rothe* (Ann Arbor, Mich.).

Darbo, Gabriele. Punti uniti in trasformazioni a codominio non compatto. Rend. Sem. Mat. Univ. Padova 24, 84–92 (1955).

Let Σ be a complete metric space and X a bounded subset of Σ. Let $\alpha(X)$ be the inf of positive numbers ϵ for which there exist finite decompositions of X into sets of diameter less than ϵ. A continuous mapping $\xi: \Sigma \rightarrow \Sigma$ is an α-contraction if it carries bounded sets into bounded sets and satisfies $\alpha\xi X \leqq k\alpha X$ where k is a non-negative number <1, independent of X. Theorem: An α-contraction which carries a closed convex subset of a complete normed linear metric space into itself admits at least one fixed point. The theorem is proved by applying a fixed-point theorem of Schauder [Studia Math. 2, 171–180 (1930)] to a certain subset. *P. A. Smith* (New York, N. Y.).

Weier, Josef. Abhängigkeit des Indexes von der Einbettung. Arch. Math. 6, 348–352 (1955).

Let U be a closed neighborhood of the origin O in R_n and let L be an r-dimensional subspace $(r<n)$ of R_n containing O. It is shown that there exists a mapping $U \rightarrow R_n$ having O for unique fixed point with arbitrary pre-assigned index, and agreeing on $U \cap L$ with a pre-assigned mapping $U \cap L \rightarrow L$ which has O for unique fixed point. *P. A. Smith* (New York, N. Y.).

Griffiths, H. B. A mapping theorem in "local" topology. J. London Math. Soc. 28, 269–278 (1953).

This is a sequel to three papers by the author: Proc. London Math. Soc. (3) 3, 350–367 (1953); Michigan Math. J. 2, 61–89 (1954); and an as yet unpublished paper [MR 15, 457; 16, 159]. The main purpose is to establish a theorem concerning local homotopy groups and locally compact metric spaces as related by a certain type of mapping between such spaces. The theorem is "an analogue of the fact that spaces of the same homotopy type have the same homotopy groups." *S. S. Cairns* (Urbana, Ill.).

Griffiths, H. B. The fundamental group of two spaces with a common point: a correction. Quart. J. Math. Oxford Ser. (2) 6, 154–155 (1955).

See same Quart. (2) 5, 175–190 (1954); MR 16, 389.

Aleksandrov, P. On certain new achievements in the combinatorial topology of nonclosed sets. Fund. Math. 41, 68–88 (1954). (Russian)

Expository article. The author describes briefly the history of the problems of the combinatorial theory of nonclosed sets, beginning with the program he outlined at the 1935 topology conference in Moscow; then contributions of Kuratowski, Čech, Čogošvili, Dowker, Hemmingsen, and Miščenko are sketched. The main part of the paper is an exposition of the results of Sitnikov on dimension properties of arbitrary subsets of E^n, the invariance and duality theorems of Sitnikov and the author, and the relation between these [for most of these results cf. P. Aleksandrov, Mat. Sb. N.S. **33**(75), 241–260 (1953); MR 16, 503; and K. A. Sitnikov, ibid. **34**(76), 3–54 (1954); MR 16, 736]. Several problems of importance for topology of arbitrary subsets of E^n are formulated: Can the Sitnikov duality be expressed without using cohomology? Is there a homology characterization, based on arbitrary coverings, of dimension? Are the groups $\Delta_c{}^p(A, \mathfrak{A})$ (homology with compact carriers) topological invariants of the complement of A?

H. Samelson (Ann Arbor, Mich.).

Bokštein, M. F. On a dimensional dominant of sets. Mat. Sb. N.S. **36**(78), 311–334 (1955). (Russian)

Denote by A an arbitrary closed bounded subset of a Euclidean space, and by dim A its dimension. P. Alexandroff [Math. Ann. **106**, 161–238 (1932)] has introduced the dimension $\Delta_m A$ of A modulo m, m an integer $\geqq 2$. The author denotes by $\dim_m A$ the dimension of A with respect to the dominant m, and has proved [Fund. Math. **34**, 311–315 (1947); MR 10, 56] that $\dim A = \max \dim_m A$. The (prime) dominant of a set A is the set of all (prime) m with $\dim_m A = \dim A$; the (prime) subdominant the set of all (prime) m with $\Delta_m A = \dim A$. Previous results of the author show that the prime dominant determines the dominant, and the prime subdominant the subdominant. In this paper, the structure of these four sets are completely characterized. It is shown that no new relations hold. More precisely, if D is any non-empty set of primes and S an arbitrary subset of D, then there exists an A whose prime dominant is D and whose prime subdominant is S.

E. E. Floyd (Charlottesville, Va.).

Švarc, A. S. On the metric order of closed sets of Euclidean space. Mat. Sb. N.S. **36**(78), 263–270 (1955). (Russian)

Suppose that A is a finite subset of a compact metric space F. If x is a point of F, the order of x relative to A is the number of points of A at a minimum distance to x. The order of A relative to F is the maximum of the orders of points x relative to A. The space F is of order n if for each $\epsilon > 0$ there is an ϵ-net (a finite set, ϵ-dense in F) in F of order n and if for some $\epsilon > 0$, every ϵ-net is of order $\geqq n$. If $\omega(F)$ is the metric order, then $\omega(F) \geqq \dim F + 1$. F is metrically correct if equality holds. The purpose of the paper is to prove two theorems. (1) Every dimensionally homogeneous $(n-1)$-dimensional curved polyhedron in n-space E^n is metrically correct. (2) If $F \subset E^{2n+2}$ and F is a compact space of dimension n, then there is an isotopy f_t of F in E^{2n+2} such that each $f_t(F)$, $0 < t \leqq 1$, is metrically correct. These theorems are related to theorems of Sitnikov [Dokl. Akad. Nauk SSSR (N.S.) **67**, 229–232 (1949); MR 11, 121].

E. E. Floyd (Charlottesville, Va.).

***Adem, José. Algebraic operations in topology and some applications to geometrical problems.** Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 179–189. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

In this expository address the author discusses several problems in algebraic topology. Most of the results mentioned have already been published in previous papers by the author. *W. S. Massey* (Providence, R. I.).

Frenkel, Jean. Cohomologie à valeurs dans un faisceau non abélien. C. R. Acad. Sci. Paris **240**, 2368–2370 (1955).

For any sheaf $F = \{F_x\}$ $(x \in X)$ of sets over a space X, $H^0(X, F)$ denotes the totality of sections of F. If F is a sheaf of groups (in general, not abelian), then $H^0(X, F)$ has a group structure, and $H^1(X, F)$ can be defined in a manner similar to the classical case. Suppose A, B to be two sheaves of groups over X, and A a subsheaf of B. The sets $\{B_x/A_x\}$ $(x \in X)$ define a sheaf B/A of homogeneous spaces over X. In the natural way, the group $H^0(X, B)$ operates on the set $H^0(X, B/A)$ of sections of the sheaf B/A.

Just as in the case of sheaves of abelian groups, there is a sequence

$$\text{(I)}\quad e \to H^0(X, A) \to H^0(X, B) \to H^0(X, B/A) \xrightarrow{\delta} H^1(X, A) \to H^1(X, B) \to H^1(X, B/A)$$

with the understanding that the last term $H^1(X, B/A)$ should be omitted if A is not distinguished in B (i.e., not all A_x are normal subgroups of B_x). Though $H^0(X, B/A)$, $H^1(X, A)$, $H^1(X, B)$, $H^1(X, B/A)$ are, in general, not groups, yet each of them has a distinguished element which will be called the neutral element. The author states the following two theorems: (i) The sequence (I) is exact in the sense that the inverse image of the neutral element under a mapping coincides with the image of the preceding mapping. Two elements of $H^0(X, B/A)$ have the same image under δ if they belong to the same class of intransitivity under the group $H^0(X, B)$. (ii) If A is abelian and distinguished in B, then δ is a semi-crossed homomorphism. If moreover X is paracompact, then a necessary and sufficient condition for an element z of $H^1(X, B/A)$ to belong to the image of $H^1(X, B)$ is that $\Delta(z) = 0$, where Δ is a map defined on $H^1(X, B/A)$.

Applications of these two theorems to problems concerning fibre spaces are also discussed.

A majority of the results are also proved in Dedecker's paper reviewed below. *H. C. Wang.*

***Dedecker, Paul. Extension du groupe structural d'un espace fibré.** Colloque de topologie de Strasbourg, mai 1955, 15 pp. Institut de Mathématique de l'Université de Strasbourg, 1955.

This paper is mainly concerned with the extension of the structure group of a principal fibre space. As a tool, the author discusses the cohomology theory of a space with coefficients in a sheaf of groups (not necessarily abelian). Let $\mathcal{G}$ be a sheaf of groups over a paracompact space B; $H^0(\mathcal{G})$ and $H^1(\mathcal{G})$ are defined. The former forms a group while the latter, in general, does not. We shall call, the

cohomology class consisting of the coboundaries, the neutral element. It is proved that an exact sequence

$$0\to\mathfrak{N}\to\mathfrak{G}\to\mathfrak{H}\to 0$$

of sheaves of groups over B induces an exact sequence

$$\text{(I)}\quad 0\to H^0(\mathfrak{N})\to H^0(\mathfrak{G})\to H^0(\mathfrak{H})\to H^1(\mathfrak{N})\to H^1(\mathfrak{G})\to H^1(\mathfrak{H}),$$

where "exact" is used in the sense that the image of each mapping coincides with the inverse image of the neutral element under the succeeding mapping. Suppose, moreover, that $i(\mathfrak{N})$ belongs to the center of $\mathfrak{G}$. Then $H^2(\mathfrak{N})$ has a meaning and a natural mapping δ_1^*: $H^1(\mathfrak{H})\to H^2(\mathfrak{N})$ is defined. If we extend the above sequence (I) by this mapping δ_1^*, the extended sequence is also exact. Using this sequence, the author proves the following: Let G be a topological group, N is a closed central subgroup of G, $H=G/N$, and E a principal fibre space (locally trivial) over a paracompact space B with structure group H. Denote by $\mathfrak{N}$, $\mathfrak{H}$ the sheaves of germs of continuous mappings of B into N, H respectively, and by γ the element in $H^1(B, \mathfrak{H})$ corresponding to E. Suppose that H has a local cross-section in G. Then the structure group H of E can be extended to G if and only if $\delta_1^*(\gamma)=0$.

Most of the results here are also proved by J. Frenkel in the paper reviewed above. *H. C. Wang* (Seattle, Wash.)

Hirsch, Guy. Sur les groupes d'homologie des espaces fibrés. Bull. Soc. Math. Belgique **6** (1953), 79–96 (1954).

Le résultat essentiel est le suivant. Soit E un espace fibré de base B connexe (par arcs), de fibre F. Soit K un corps de coefficients. Faisons les hypothèses: (I) pour tout entier $q\geq 0$, l'espace de cohomologie $H^q(F; K)$ est un K-espace vectoriel de dimension finie; (II) pour tout q, le groupe fondamental $\pi_1(B)$ opère trivialement dans $H^q(F; K)$. Alors il existe une application K-linéaire U de $H^*(F; K)$ dans l'espace vectoriel $C^*(E; K)$ des cochaînes de E, conservant le degré, et jouissant des deux propriétés suivantes: (a) l'application composée $H^*(F; K)\xrightarrow{U}C^*(E; K)\to C^*(F; K)$ applique chaque $x\in H^*(F; K)$ sur un cocycle de la classe de x; (b) si on prolonge U en $\bar{U}$:

$$C^*(B; K)\otimes_K H^*(F; K)\to C^*(E; K)$$

en posant $\bar{U}(b\otimes x)=f(b)\smile U(x)$, où f: $C^*(B; K)\to C^*(E; K)$ désigne l'homomorphisme défini par la projection $E\to B$, et où $\smile$ désigne le cup-produit, alors l'application $\bar{U}$ a un noyau nul, et son image est stable pour le cobord δ de $C^*(E; K)$. Il s'ensuit que si on transporte à

$$C^*(B; K)\otimes H^*(F; K)$$

l'opérateur δ de $C^*(E; K)$ au moyen de $\bar{U}$, $\bar{U}$ définit un isomorphisme des cohomologies

$$H^*(C^*(B; K)\otimes H^*(F; K))\approx H^*(E; K).$$

Ainsi la cohomologie de l'espace fibré E peut être calculée à l'aide d'un opérateur différentiel convenable sur

$$C^*(B; K)\otimes H^*(F; K).$$

L'article ne donne pas de démonstrations détaillées. L'auteur dit qu'il suppose que B est un polyèdre de dimension finie, mais que le théorème est valable pour tous les espaces fibrés au sens de Serre (il faut alors prendre pour $C^*(B; K)$ et $C^*(E; K)$ les espaces de cochaînes singulières). L'auteur fait intervenir une filtration de $H^*(F; K)$ qui n'a pas de caractère intrinsèque, et dont la signification échappe au rapporteur.

Note du rapporteur: non seulement le théorème est valable pour tous les espaces fibrés au sens de Serre, mais on peut le préciser: U peut être choisie de manière que l'opérateur δ défini sur $C^*(B; K)\otimes H^*(F; K)$ grâce à $\bar{U}$ applique $C^*(B; K)\otimes H^q(F; K)$ dans

$$C^*(B; K)\otimes \sum_{n<q} H^n(F; K).$$

En outre, le théorème est encore vrai sans l'hypothèse (I), à condition de remplacer $C^*(B; K)\otimes_K H^*(F; K)$ par $\mathrm{Hom}(C_*(B), H^*(F; K))$, $C_*(B)$ désignant le groupe des chaînes singulières de la base B. Enfin, le théorème est valable même si K n'est pas un corps, pourvu que le K-module $H^*(F; K)$ possède une K-base homogène.

H. Cartan (Paris).

Shimada, Nobuo, and Uehara, Hiroshi. Some remarks on Adem's extension theorem. Mem. Fac. Sci. Kyûsyû Univ. Ser. A. **9**, 37–46 (1955).

The authors show how to modify the operation ϕ_0 introduced by the reviewer [Proc. Nat. Acad. Sci. U. S. A. **38**, 720–726 (1952); MR **14**, 306] so as to obtain two new operations ϕ_2 and ϕ_4. All these are cohomology operations of second order. They are defined on the kernel of Sq^2 and have values on some factor groups of cohomology groups; ϕ_0 is defined on integral classes with modulo 2 values, ϕ_2 is defined on modulo 2 classes with modulo 4 values, ϕ_4 is defined on modulo $4r$ classes with modulo 2 values. Using the Steenrod squares and these operations, the authors give a formula expressing the third obstruction when mapping a complex into an $(n-1)$-connected space ($n>2$). Consequently, the classification of homotopy classes of maps of an $(n+2)$-complex into an $(n-1)$-connected space can be obtained by standard methods.

J. Adem.

Errera, Alfred. Sur la classification des polyèdres de genre zéro. Bull. Soc. Math. Belg. **1951**, 51–66 (1952).

The polyhedra considered are such that exactly three faces meet at every vertex and fewer than four faces do not form an annulus; these polyhedra are called normal. According to a theorem of H. Whitney [Ann. of Math. (2) **32**, 378–390 (1931)] on any normal polyhedron there can be drawn at least one simple closed contour which avoids all the vertices and enters every face just once. Such a simple closed contour decomposes the graph formed by the edges of the polyhedron into two trees. The interior vertices of these trees are of degree 3, and each tree has f end vertices where f denotes the number of the faces of the polyhedron. Conversely, a polyhedron (not necessarily normal) can be obtained from two such trees by selecting one end vertex in each tree and joining them and then joining the end vertices of one tree to the end vertices of the other in cyclic order. This can be done in f different ways. The enumeration of all the possible polyhedra is attempted by means of this principle.

G. A. Dirac (Vienna).

Leech, John. Seven region maps on a torus. Math. Gaz. **39**, 102–105 (1955).

It is well known that the surface of a torus can be divided into seven simply connected regions each of which abuts on to each of the others. The various ways of constructing these configurations are here analysed.

G. A. Dirac.

GEOMETRY

Chisini, O. Sulla non dimostrabilità del postulato di Euclide. Period. Mat. (4) **33**, 65–74 (1955).

Thébault, Victor. Sur quelques inégalités relatives aux rayons des cercles exinscrits à un triangle. Mat. Lapok **3**, 59–61 (1952). (Hungarian. Russian and French summaries)

✶Monographs on topics of modern mathematics relevant to the elementary field. Edited by J. W. A. Young. Dover Publications, Inc., New York, 1955. xvi+416 pp. Clothbound: $3.95; paperbound: $1.90.

Photo-offset reprint, with a new introduction by Morris Kline, of the first edition [Longmans, Green, New York, 1911]. The individual monographs are: The foundations of geometry, by O. Veblen; Modern pure geometry, by T. F. Holgate; Non-Euclidean geometry, by F. S. Woods; The fundamental propositions of algebra, by E. V. Huntingdon; The algebraic equation, by G. A. Miller; The function concept and the fundamental notions of the calculus, by G. A. Bliss; The theory of numbers, by J. W. A. Young; Constructions with rule and compass; regular polygons, by L. E. Dickson; and The history and transcendence of π, by D. E. Smith.

Jaswon, M. A., and Dove, D. B. The geometry of lattice planes. I. Acta Cryst. **8**, 88–91 (1955).

Consider a lattice of vector end points for vectors in the form $ua+vb+wc$, where a, b, and c are linearly independent vectors and u, v and w are integers. The authors determine the geometrical arrangements of lattice points lying on a plane $hu+kv+lw=m$ where h, k, l and m are integers by specifying primitive translation vectors for the lattice points in this plane. In general, obtaining these translation vectors involves an examination of a number of possibilities. The case of a translated lattice, based on a "motif point" is also treated. *F. J. Murray* (New York, N. Y.).

Locher-Ernst, L. Konstruktionen des Dodekaeders und Ikosaeders. Elem. Math. **10**, 73–81 (1955).

Singal, M. K., and Bhatnagar, P. L. A problem on moments. Math. Student **22** (1954), 167–174 (1955).

The distance z to a hyperplane E_{n-1} of the centre of gravity of a simplex, the density at any point of which varies as the pth power of the distance of the point to E_{n-1} is evaluated. The result is

$$z=(p+1)S_{n+1,\,p+1}/(p+1+n)S_{n+1,\,p},$$

where $S_{n+1,\,i}$ is the sum of all the different homogeneous products of $u_1, u_2, \ldots, u_{n+1}$ of ith degree ($u_1, u_2, \ldots, u_{n+1}$ being the distances to E_{n-1} of the corners of the simplex). Some particular cases are discussed. *L. A. Santaló.*

Meynieux, R. Sur le quadrilatère de Dixon et Morton déterminé par trois couples associés de trièdres de Steiner relatifs à une surface cubique. Publ. Sci. Univ. Alger. Sér. A. **1** (1954), 183–195 (1955).

The 27 lines on the general cubic surface can be regarded (in forty ways) as three sets of nine, each ennead consisting of the lines of intersection of the planes of two (Steiner) trihedra. It was proved by A. L. Dixon and V. C. Morton [Proc. London Math. Soc. (2) **37**, 221–240 (1934), p. 229] that the vertices of the six trihedra are the vertices of a complete quadrilateral in a certain plane. The author gives a new proof of this interesting result, using a certain skew-symmetric function of three points on a plane cubic curve (the section of the cubic surface by an arbitrary plane). *H. S. M. Coxeter* (Toronto, Ont.).

Byušgens, S. S. The configuration of A. K. Vlasov and its generalization. Mat. Sb. N.S. **36(78)**, 275–280 (1955). (Russian)

Denote the n-dimensional projective space by P^n and by L^r a linear subspace of dimension r. Vlasov's theorem states: Let $\alpha_1, \ldots, \alpha_4$ be L^2 in P^4 no two of which lie in one L^3, so that $\alpha_i \cap \alpha_j$ is a point p_{ij}. For any three α_i there is exactly one L^2 intersecting each of the three α_i in a line. Choose the notation β_k for the L^2 such that β_i contains the three p_{ij} not contained in α_k. Unless the β_i coincide, the 4 points $\alpha_k \cap \beta_k$ lie in an L^2.

A simple proof, avoiding quadrics, is given and the theorem is extended to m spaces L^k in P^n, under appropriate conditions on m, k, n, non-degeneracy of the intersections, etc. Both these conditions and the final theorem are too involved to be formulated here. *H. Busemann.*

Kroch, Aryeh. Collineation of space by erection. Riveon Lematematika **8**, 65–75 (1954). (Hebrew. English summary)

Discussion of the decomposition of collineations in 3-space into orthogonal transformations, projections and erections. (If P_1 and P_2 are projections of the variable point P on two planes, then the mapping such that P is the image of the pair P_1, P_2 is called an erection.) *T. S. Motzkin.*

✶Tits, J. Collinéations et transitivité. III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 66–67. Fédération belge des Sociétés Scientifiques, Bruxelles.

A proper projective space of n dimensions has as points ordered $(n+1)$-tuples of elements of an arbitrary commutative field K, not all zero, with $(x_1, x_2, \ldots, x_{n+1})$ and $(y_1, y_2, \ldots, y_{n+1})$ identified if and only if $y_i=k\cdot x_i$ $(i=1, 2, \ldots, n+1)$, $0\neq k \in K$. Transformations $y_i=x_ja_i^j$, $|a_i^j|\neq 0$, are collineations, the group of which forms the projective group of the space. A general projective space of n dimensions is formed by the elements of an abstract set, certain subsets of which (called linear subspaces of dimensions $-1, 0, \ldots, n$) are the elements of a lattice with set inclusion as partial ordering relation. The author considers an almost n-tuply transitive group G of projective type [Colloq. Internat. Centre Nat. Rech. Sci., Paris, no. **24**, 207–208 (1950); MR **13**, 9] operating in such a space and shows, for example, that the space is proper projective provided those transformations of G for which each of three points of a line is invariant leave the line pointwise fixed. *L. M. Blumenthal* (Columbia, Mo.).

Castrucci, Benedito. Fundamental postulates of projective geometry. Soc. Parana. Mat. Annuário **1**, 1–9 (1954). (Portuguese)

A brief outline of the standard axiomatic approach. *H. Busemann* (Los Angeles, Calif.).

✶Coxeter, H. S. M. The real projective plane. 2d ed. Cambridge, at the University Press, 1955. xii+226 pp. $4.75.

This edition differs from the first [Cambridge, 1949; MR **10**, 729] in correction of small errors and new treatment of several sections.

Dempster, A. P., and Schuster, S. Constructions for poles and polars in n-dimensions. Pacific J. Math. **5**, 197–199 (1955).

The authors give, in projective n-space, a linear construction for the polar hyperplane of an arbitrary point with respect to the symmetric polarity in which a given simplex is self-polar while a given point and hyperplane are pole and polar. When $n=2$ this is the construction 5.64 of Coxeter, The real projective plane [2nd ed.; reviewed above].
H. S. M. Coxeter (Toronto, Ont.).

Court, N. A. Pencils of conics. Bol. Mat. **28**, no. 2, 6–15 (1955).

The author gives synthetic solutions for various projective problems, e.g., he finds the locus of the trilinear pole of a variable line through a fixed point [cf. Coxeter, The real projective plane, 2nd ed., Cambridge University Press, 1955, pp. 114, 185 (Ex. 2), 205; reviewed second above].

The pencil of conics $c_1x_1^2+c_2x_2^2+c_3x_3^2=0$, $c_1+c_2+c_3=0$ [Coxeter, op. cit., p. 209] and the range of conics

$$C_1X_1^2+C_2X_2^2+C_3X_3^2=0, \quad C_1+C_2+C_3=0,$$

are said to be "harmonically associated", since their basic quadrangle $(1, \pm1, \pm1)$ and basic quadrilateral $[1, \pm1, \pm1]$ are so situated that each side of the latter is the trilinear polar of the corresponding vertex of the former with respect to the triangle formed by the remaining three vertices (and also with respect to the common diagonal triangle, which is the triangle of reference). The conic $\sum C_iX_i^2=0$ touches the line $[1, 1, 1]$ at (C_1, C_2, C_3), which lies on $\sum c_ix_i^2=0$ if $\sum c_iC_i^2=0$. This relation establishes a (1, 2) correspondence between conics of the pencil and conics of the range. For a given conic $\sum c_ix_i^2=0$ ($\sum c_i=0$) of the pencil, the two corresponding conics $\sum C_iX_i^2=0$ of the range touch the four lines $[1, \pm1, \pm1]$ at the two tetrads of points $(C_1, \pm C_2, \pm C_3)$ which, with the four basic points $(1, \pm1, \pm1)$, all lie on the conic $\sum c_ix_i^2=0$.

Since $\sum C_i=0$, the trilinear polar of the point of contact (C_1, C_2, C_3) envelops the conic $X_2X_3+X_3X_1+X_1X_2=0$ or $x_1^{1/2}\pm x_2^{1/2}\pm x_3^{1/2}=0$ [Coxeter, op. cit., p. 206 (12.75)], which is inscribed in the triangle of reference and circumscribed about the triangle $(0, 1, 1)(1, 0, 1)(1, 1, 0)$. This conic is also the envelope of the polar of $(1, 1, 1)$ with respect to the variable conic $\sum C_iX_i^2=0$. By taking $[1, 1, 1]$ to be the line at infinity, the author deduces an affine theorem due to A. J. Pressland: The polar of the centroid of a triangle with respect to any escribed parabola is tangent to the circumscribed ellipse of minimum area.
H. S. M. Coxeter (Toronto, Ont.).

✶Bilo, M. J. Sur le théorème fondamental (au sens restreint) de la géométrie projective quaternionienne. III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 93–96. Fédération belge des Sociétés Scientifiques, Bruxelles.

The author clarifies an earlier paper [Simon Stevin **28**, 140–145 (1951); MR **13**, 487] by means of an analytic treatment. When the point with abscissa $t+xi+yj+zk$ on the quaternion projective line is represented by the point (t, x, y, z) in real conformal 4-space, the 1-chains, 2-chains and 3-chains appear as circles, 2-spheres and 3-spheres. He shows that the condition for the equivalence of two tetrads of points on the quaternion line is the equality of the norms of their cross-ratios taken in two different orders, in agreement with a result of Study [Math. Z. **21**, 45–71 (1924), p. 67] for the conformal equivalence of two tetrads of real non-concyclic points. *H. S. M. Coxeter* (Toronto, Ont.).

Pírko, Zdeněk. The harmonic correspondence. Časopis Pěst. Mat. **76**, 201–215 (1951). (Czech)

Dieser Aufsatz knüpft an eine frühere Untersuchung analagmatischer Kurven und quadratischer Inversionen [Časopis Pěst. Mat. Fys. **75**, D266–D276 (1950)]. Unter geeigneten Voraussetzungen über die Parameterfunktionen $\lambda_i=\lambda_i(t)$ bildet die einparametrige Schar von Kegelschnitten

$$(*) \qquad \lambda_1(t)x_2x_3+\lambda_2(t)x_1x_3+\lambda_3(t)(x_1x_2+x_3^2)=0$$

ein analagmatisches Netz. Von besonderer Bedeutung ist dabei die Einhüllende dieser Schar. Der Punkt $\lambda_1:\lambda_2:-\lambda_3$ heisst Mittelpunkt der Schar (*). In ihrer Abhängigkeit von t bestimmen dann die Verhältnisgleichungen

$$y_1:y_2:y_3=\mu_1(t):\mu_2(t):\mu_3(t)=\lambda_1(t):\lambda_2(t):-\lambda_3(t)$$

die sogenannten "Deferenten" der analagmatischen Kurven. Die eineindeutige Korrespondenz zwischen beiden drückt sich dabei durch eine weitere Verwandtschaft aus, in welcher aber einem Punkt nicht wieder ein Punkt sondern eine Gerade entspricht. Verfasser nennt diese Korrespondenz harmonische Verwandtschaft und gibt ihre analytische Darstellung an. Diese harmonische Verwandtschaft erweist sich von birationalem Charakter. Auch ihre Ausartungen (singuläre Korrelationen) werden erwähnt. *M. Pinl.*

Pírko, Zdeněk. The harmonic correspondence. II. Časopis Pěst. Mat. **79**, 261–272 (1954). (Czech)

Im Anschluss an die vorhergehende Untersuchung (vgl. vorstehendes Referat) behandelt Verfasser nunmehr gewisse spezielle Eigenschaften, der im ersten Aufsatz definierten harmonischen Verwandtschaft **H**. Zunächst werden die aus der Theorie birationaler Punkttransformationen bekannten Begriffe, selbstkonjugierter Punkte, selbstkonjugierter Geraden (allgemein selbstkonjugierter Kurven) auf die Zuordnungen der Verwandtschaft **H** und ihrer inversen $\mathbf{H}^{-1}$ übertragen und Bedingungen für solche Punkte und Geraden angegeben. Zur harmonischen Verwandtschaft **H** gibt es nur die folgenden selbstkonjugierten Gebilde: die Ecken des Basisdreiecks und das Kegelschnittsbüschel, dessen Kurven beide Seiten des Basisdreiecks berühren, die vom Scheitel P zu den übrigen beiden Scheiteln führen. Die weitere Untersuchung verknüpft nunmehr die harmonische Verwandtschaft **H** mit einer Polarverwandtschaft **P** und einer quadratischen Inversion **I** und führt auf die Diskussion der symbolischen Produkte **PI**, **PH**, **HI** usw. Als wichtige Sonderfälle werden weiterhin spezielle Grundkurven gewählt, wie Lamé's Kurve

$$\sigma_1x_1^n+\sigma_2x_2^n+\sigma_3x_3^n=0 \quad (\sigma_i, n \text{ Konstante}),$$

oder die W-Kurve

$$x_1^{\sigma_1}x_2^{\sigma_2}x_3^{\sigma_3}-k=0 \quad (\sigma_i, k \text{ Konstante}; \sigma_1+\sigma_2+\sigma_3=0).$$

Schliesslich werden Fälle autopolarer oder analagmatischer Grundkurven behandelt. *M. Pinl* (Köln).

Varga, Ottó. L'influence de la géométrie de Bolyai-Lobatchevsky sur le développement de la géométrie. Acta Math. Acad. Sci. Hungar. **5**, supplementum, 71–94 (1954). (Russian summary)

[This paper appeared earlier in Hungarian in Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 151–171 (1953); MR **15**, 383.] It discusses briefly the foundations of geometry by Hilbert, the role of Archimedes' axiom, the Cayley-Klein projective metrics, foundations of geometry based on groups

of motions, Riemannian geometry, the geometry of paths, Finsler spaces considered as spaces of line elements.

The choice of material is highly personal. To find Bolyai's influence in the last subject seems far-fetched, whereas one misses many names or investigations, which doubtless belong here, most of all Hjelmslev's work; the group-theoretical approach looks as though it had stopped with Hilbert's 1902 papers [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1902**, 233–241; Math. Ann. **56**, 381–422 (1902)].

H. Busemann (Los Angeles, Calif.).

Coxeter, H. S. M. Regular honeycombs in elliptic space. Proc. London Math. Soc. (3) **4**, 471–501 (1954).

Arrangements of equal regular polyhedral cells fitting together to fill an elliptic 3-space are called honeycombs. The six regular honeycombs correspond to the projections of the six regular polytopes in Euclidean 4-space onto the circumscribed 3-sphere, in which antipodal points are identified to define an elliptic 3-space. Those with 5, 4, 8, 12, 60 and 300 vertices are denoted respectively by α, β, γ, δ, ϵ and ζ. The honeycomb β consists of 8 totally rectangular tetrahedra, each bounded by the same set of four planes, and γ is its dual. Likewise the 60 dodecahedra of ζ form the dual of the 300 tetrahedra of ϵ. S. L. van Oss described a continuous mapping of the rotations P, Q of elliptic space about a fixed point E onto correspondingly lettered points P, Q, etc. of the same space so that the distance between P and Q is half the angle of the rotation $P^{-1}Q$. The displacement that maps each point P of the elliptic space into a corresponding point $Q^{-1}PR$ for fixed Q, R is said to be the product of a left screw Q_l and a right screw R_r, and every displacement has a unique representation of this form. To the elements of the finite rotation groups correspond symmetrical arrangements of points in elliptic space: e.g. the 4 operations of the four-group D_4 are vertices of a β, and the remaining 8 operations of order 3 in a tetrahedral group T_{12} that contains D_4 form a γ, whereas together the 12 points of T_{12} form a δ. Also the 12 odd permutations of the octahedral group form a congruent δ', obtained from the first by a right screw, and reciprocal to δ. The 60 operations of the icosahedral group I_{60} correspond to vertices of an ϵ, and the 300 even permutations of 0, 1, 2, 3, 4, 5 that displace 0 correspond to vertices of a ζ. Thus if an even permutation of 0, 1, 2, 3, 4, 5 takes the digits 1, 2, 3, 4, 5 into a, b, c, d, e respectively, the symbol $abcde$ may be used to represent a vertex of a ζ or its reciprocal ϵ according as 0 does or does not appear. Distances between points are then read directly from the quotient of the corresponding permutations. The rows of Latin squares of order four and five are used to study a number of subconfigurations such as desmic tetrahedra, and tetrahedra inscribed in an ϵ. The vertices of ζ are classified into sets at various distances from E, and a complete description is given of the Petrie polygons of ϵ and of ζ.

J. S. Frame (East Lansing, Mich.).

***Bonola, Roberto. Non-Euclidean geometry, a critical and historical study of its developments. Translation with additional appendices by H. S. Carslaw. Supplement containing the G. B. Halsted translations of "The science of absolute space" by John Bolyai and "The theory of parallels" by Nicholas Lobachevski.** Dover Publications, Inc., New York, 1955. xii+268+xxx+71+50 pp. Clothbound: $3.95; paperbound: $1.90.

The three translations reprinted here by photo-offset are reproduced from the following editions: Open Court, Chicago, 1912; The Neomon, Austin, Tex., 1896; Open Court, Chicago, 1914.

van Heemert, A. Zur Kennzeichnung der Systeme der Kreise und der Kegelschnitte. J. Reine Angew. Math. **194**, 183–189 (1955).

In two papers Buckel characterized the system of all circles of the euclidean plane with positive or infinite radii, and the system of all non-degenerate conics of the projective plane [same J. **191**, 13–29, 165–178 (1953); MR **15**, 149]. Each of the characterization theorems assumed the arc-wise connectedness of each figure of the system. This paper weakens this assumption to connected and locally compact in the first case, and to merely connected in the second. The proofs offered are, moreover, considerably shorter than are those in the two previous papers.

L. M. Blumenthal.

Freudenthal, Hans. Die Bedeutung der topologischen Voraussetzung bei der Buckel-van Heemertschen Charakterisierung des Systems der Kreise. J. Reine Angew. Math. **194**, 190–192 (1955).

The author constructs an example to show that the requirement "connected and locally compact" made in van Heemert's characterization of the system of all circles of the euclidean plane [see the preceding review] cannot be weakened to "connected" only.

L. M. Blumenthal.

Convex Domains, Extremal Problems, Integral Geometry

Schäffer, Juan J. Smallest lattice-point covering convex set. Math. Ann. **129**, 265–273 (1955).

The following problem, proposed by Santaló, is solved: to find a closed convex set of minimal area in the cartesian plane, such that the set contains in any position at least one point with integral coordinates. The solution is unique and the same as that given by D. B. Sawyer. [Quart. J. Math. Oxford Ser. (2) **4**, 284–292 (1953); MR **15**, 607] under the assumption that the set have a center, namely a unit square with segments of parabolas added on opposite sides, where the parabolas form 45° angles with these sides. The methods of Sawyer are used, who, according to a note added in proof, in the meantime also solved the general problem.

H. Busemann (Los Angeles, Calif.).

Bambah, R. P. Polar reciprocal convex bodies. Proc. Cambridge Philos. Soc. **51**, 377–378 (1955).

Let K and K' be two convex bodies in the n-dimensional Euclidean space which are polar reciprocal to each other with respect to the unit sphere J_n centered at the origin 0. Let $V(K)$, $V(K')$ and J_n be the volumes of K, K' and J_n respectively. The author proves the inequalities: a) $V(K)V(K')\geqq n^{-n/2}J_n^2$ for K having 0 as centre of symmetry; b) $V(K)V(K')\geqq 4^n/(n!)^2$ for K having 0 as an inner point.

L. A. Santaló (Buenos Aires).

Gohier, Simone. Sur les surfaces convexes ouvertes à courbure intégrale 4π. C. R. Acad. Sci. Paris **240**, 2291–2293 (1955).

Let S be a convex surface homeomorphic to a disk such that its curvature is 2π. Then S is rigid. The special case where the spherical image of the boundary B of S reduces to a point, i.e. where the "piece missing" to complete S to a closed convex surface lies in a plane tangent to S along B,

is well known and due to Rembs. The present case is reduced to Pogorelov's general rigidity theorem. In case the "missing piece" lies on a cylinder, S is also infinitesimally rigid.

H. Busemann (Los Angeles, Calif.).

Ohmann, D. Über den Brunn-Minkowskischen Satz. Comment. Math. Helv. **29**, 215–222 (1955).

For a compact point set C in the euclidean n-space let $W_0(C)$ denote the n-dimensional measure of C. The cross measure (Quermass) of C in the direction ξ is by definition the $(n-1)$-dimensional measure of the orthogonal projection of C onto a hyperplane with normal vector ξ. Denote by $W_1(C)$ the mean value of the cross measures (the Quermass-integral) of C. (When C is convex, $W_1(C)$, apart from a constant factor, is the surface area of C.) The author gives simple proofs of the generalized Brunn-Minkowski inequality for compact sets A and B (due to Lyusternik)

$$(1)\qquad W_0(A+B)^{1/n}\geqq W_0(A)^{1/n}+W_0(B)^{1/n}$$

and of the inequality

$$(2)\qquad W_1(A+B)^{1/(n-1)}\geqq W_1(A)^{1/(n-1)}+W_1(B)^{1/(n-1)}$$

hitherto only known for convex sets. The principal lemma (due to Bonnesen in the convex case) states that if A and B have equal cross measures in some direction, then

$$W_0((1-\lambda)A+\lambda B)\geqq(1-\lambda)W_0(A)+\lambda W_0(B)\qquad(0\leqq\lambda\leqq 1).$$

From this result (1) follows easily, and (2) is obtained by generalizing Bonnesen's proof for convex sets [see T. Bonnesen and W. Fenchel, Theorie der konvexen Körper, Springer, Berlin, 1934, pp. 107–108]. If $A-B$, that is, the set of all points x such that $x+B\subseteq A$, is non-empty, (1) and (2) yield immediately

$$(1')\qquad W_0(A-B)^{1/n}\leqq W_0(A)^{1/n}-W_0(B)^{1/n},$$

$$(2')\qquad W_1(A-B)^{1/(n-1)}\leqq W_1(A)^{1/(n-1)}-W_1(B)^{1/(n-1)}.$$

The author constructs examples showing that each of the systems (1), (1′) and (2), (2′) is a complete system of inequalities for the 4 quantities $W_p(A)$, $W_p(B)$, $W_p(A+B)$, $W_p(A-B)$, $p=0$ or 1.

W. Fenchel (Copenhagen).

Algebraic Geometry

Benedicty, Mario. Sur une généralisation de la notion d'équivalence linéaire sur une courbe algébrique. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41**, 551–555 (1955).

Soit K un corps de fonctions algébriques d'une variable sur k, et A un sous anneau de K, contenant k et admettant K pour corps des fractions. Deux diviseurs D, D' de K sont appelés A-équivalents si $D-D'$ est le diviseur d'un élément inversible de A. L'analogue du théorème de Riemann n'est pas nécéssairement vrai si on remplace l'équivalence linéaire (cas $A=K$) par la A-équivalence. Rosenlicht a montré que, pour que ce théorème soit vrai, il suffit que l'anneau A soit semi-local [Ann. of Math. (2) **56**, 169–191 (1952); MR **14**, 80]; des exemples montrent que cette condition n'est pas nécéssaire.

P. Samuel (Mexico, D. F.).

Zappa, Guido. Sopra una probabile diseguaglianza tra i caratteri invariantivi di una superficie algebrica. Rend. Mat. e Appl. (5) **14**, 455–464 (1955).

La formule dont il est question dans le titre est la suivante: (1) $\rho+\rho_0\geqq 4p_g+1$; où p_g est le genre géométrique d'une surface algébrique F et ρ, ρ_0 sont les nombres des cycles indépendants à deux dimensions algébriques et transcendants sur F. A cause de la formule de Picard-Alexander, la relation (1) équivaut à l'autre: $I+1\geqq 4p_a$; où I est l'invariant de Zeuthen et Segre; à cause de la formule de Noether elle est aussi équivalente à $\omega\leqq 8p_a+10$; où ω est le genre linéaire de F. On trouve encore qu'elle est aussi équivalente à l'une ou à l'autre des deux formules suivantes: $\delta-m+1\geqq\chi/6$; $\delta+2p\geqq k/3$; où δ est la classe de la courbe Δ que l'one obtient comme courbe de diramation en projetant F d'un point sur le plan, m est l'ordre de F, p est le genre des sections planes de F, χ et k sont les nombres des points d'inflexion et des points de rebroussement de Δ. Sous la dernière de ces formes, la relation (1) peut être démontrée, par un passage à la limite, pour toutes les surfaces F qui peuvent se transformer par continuité en un système de plans satisfaisant à certaines hypothèses de généralité et de simplicité qu'il serait trop long d'expliquer ici. La même relation est vraie en particulier pour toutes les surfaces réglées. L'A. pense que la (1) soit vraie pour toute surface algébrique; il donne quelques renseignements sur une méthode que l'on pourrait suivre pour la démontrer en général.

E. G. Togliatti (Gênes).

Burniat, Pol. Sur un lemme de F. Enriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41**, 97–100 (1955).

Il s'agit du lemme utilisé par F. Enriques à p. 129 de son traité sur "Le superficie algebriche" [Zanichelli, Bologna, 1949; MR **11**, 202] pour passer du théorème de Riemann-Roch qui fournit la dimension d'un système linéaire de courbes d'une surface algébrique plus ample que le système canonique au même théorème pour un système quelconque. Ce lemme affirme que la série linéaire découpée par un système linéaire complet $|C|$ quelconque sur une courbe irréductible d'un système $|D|$ est complète lorsque $|D|$ est "normalement grand" par rapport à $|C|$; cela signifie à quelques détails près que la différence $|D-C|$ est irréductible et égale à $|sL|$, où $|L|$ est un système irréductible donné et s est un nombre entier arbitrairement grand. En suivant le même ordre d'idées de la démonstration de F. Enriques, l'A. y introduit quelques modifications, qui montrent que la démonstration même ne requiert que la notion d'équivalence linéaire sur une surface.

E. G. Togliatti (Gênes).

Conforto, Fabio. Complemento ad una ricerca sopra i sistemi lineari di integrali semplici di prima specie con periodi ridotti sopra una varietà di Picard. Boll. Un. Mat. Ital. (3) **9**, 119–125 (1954).

F. Conforto [Arch. Math. **5**, 282–291 (1954); MR **16**, 397] ha costruito un esempio di matrice di Riemann ω, di genere $p=3$, che possiede un sistema irregolare $\Sigma_{1;3}$ costituito da un differenziale picardiano di prima specie con tre periodi, e cioè tale che i suoi periodi si possono esprimere come combinazioni lineari a coefficienti interi di tre (e non meno) quantità complesse. Ma per ragioni di spazio ha soltanto enunciato il fatto che per una matrice di Riemann di genere 3 l'esistenza d'un siffatto $\Sigma_{1;3}$ è una effettiva particolarità; a ciò è appunto dedicata la Nota presente, redatta da M. Rosati sulla traccia di indicazioni avute da F. Conforto (deceduto a Roma il 24 febbraio 1954).

D. Gallarati (Genova).

Burau, Werner. Kennzeichnung der einfachsten Punktmodelle für die linearen Räume auf Hyperquadriken und Ausreduktion der Spinorenrelationen. Rend. Mat. e Appl. (5) **14**, 465–486 (1955).

In the following let $a=\binom{k+1}{2}$. The Grassmannian $\Gamma_{2k,k}$ of the ∞^a S_k's of one system on the general quadric Q_{2k} of

S_{2k+1}, is the complete section by a certain S_r, where

$$r=\frac{1}{2}\binom{2k+2}{k+1}-1=\binom{2k+1}{k}-1,$$

of the Grassmannian $G_{2k+1,k}$ of all S_k's of S_{2k+1}. (That of the S_k's of the other system is the section by another S_r skew to this, and spanning with it the whole ambient of $G_{2k+1,k}$.) $\Gamma_{2k,k}$ is identical with the Grassmannian $\Gamma_{2k-1,k-1}$ of all S_{k-1}'s on the general quadric Q_{2k-1} of S_{2k}. Here $\Gamma_{2k,k}$ is the V^2 transform (i.e. projective model of all quadric sections) of a variety M_a, normal in S_{2^k-1}, which is its minimal model, and received some preliminary study in Colloq. Math. 5, 4–118 (1952) [MR 15, 895]. Here many detailed properties of M_a are given, especially of the subvarieties which represent various subsystems of the S_k's on Q_{2k}, and its duality properties; for $k\equiv3$ (mod 4) it is self-polar with regard to a particular quadric on which it lies, for $k\equiv1$ (mod 4) with regard to a particular null-system; for even values of k, while it is self-dual, there is no special involutory duality. (For $k=1, 2, 3$, S_a is a line, an S_3, and a general quadric Q_6 of S_7.) M_a is the complete intersection of

$$\binom{2^k+1}{2}-\frac{1}{2}\binom{2k+2}{k+1}$$

quadrics in its ambient, whose equations are E. Cartan's "spinor-relations" [Leçons sur la théorie des spineurs, t. I, II, Hermann, Paris, 1938]. The group of projective transformations of Q_{2k} into itself induces projective groups in the ambient of M_a (in which it is simple) and in the space of the quadrics through the latter, in which its invariant spaces are studied. *P. Du Val* (London).

Stein, E. The special primal of a linear plane complex in S_n (n even) and a resulting property of certain determinantal manifolds. Rend. Mat. e Appl. (5) 14, 542–549 (1955).

With regard to a general linear complex of planes in S_n, say $a_{ijk}p_{ijk}=0$, where p_{ijk} are the Grassmann coordinates of a plane and the coefficients a_{ijk} are skew symmetric, a general line l_{ij} determines a hyperplane $(a_{ijk}l_{ij})x_k=0$, generated by planes of the complex through the line; the line is called special if $a_{ijk}l_{ij}=0$, i.e. if every plane through it belongs to the complex. A general point x_i determines a linear complex $c_{ij}l_{ij}$ $(=(a_{ijk}x_k)l_{ij})=0$ of lines joined to the point by planes of the complex. If $n=2r$ (resp., $2r+1$) and c_{ij} is of rank $2(r-h)$, the point is called special of type h; in this case the special lines through it generate an S_{2h} (resp., S_{2h+1}).

It is shewn that for even values of n, the minor of c_{ij} in $|c_{ij}|$ is of the form $x_ix_j\varphi^2$, where φ is a polynomial of degree $\frac{1}{2}(n-2)$; $\varphi=0$ is the locus $V_{n-1}^{(n-2)/2}$ of singular points (generally of type 1), and is generated by singular lines, a pencil of them through each point.

The latter part of the paper is devoted to the loci Ω^{2s} in the space of all linear complexes of lines of S_n, whose points represent complexes with a singular S_{n-2s}. It is shewn that through a general point of the complex space there pass ∞^b, $b=\binom{n}{3}$, S_n's cutting Ω^{n-2} in a $V_{n-1}^{(n-2)/2}$ similar to $\varphi=0$ above; there are ∞^{d-1}, $d=\binom{n+1}{3}$, such $V_{n-1}^{(n-2)/2}$'s on Ω^{n-2}, ∞^{b+2} of them passing through a general point of Ω^{n-2}.
P. Du Val (London).

Longo, Carmello. Su alcune proprietà dei complessi lineari in S_{2r} e problemi relativi alla loro classificazione in S_8. Rend. Mat. e Appl. (5) 14, 510–524 (1955).

This paper refers to that reviewed above and shows first that for $n=2r$ the tangent hyperplane to V_{2r-1}^{r-1} at any point P coincides with the hyperplane $a_{ijk}l_{ij}=0$ determined by any nonsingular line l_{ij} in the plane generated by the singular lines through P. The rest of the paper is devoted to the case $n=8$. It is shewn that a linear complex $L_8^{(2)}$ of planes in S_8 will have some singular points of type ≥2 (from a simple count of conditions). If there are more than one of type 2, then either there are exactly two (the line joining them being singular), or exactly 4, the lines joining them not being singular), or a line of such points; in this last case the S_4's generated by singular lines through these points generate a quadric cone whose vertex is a plane through the line. Finally, $L_8^{(2)}$'s having a singular point of type 3 are classified into five main categories with a great many subdivisions.
P. Du Val (London).

Rosenlicht, Maxwell. Automorphisms of function fields. Trans. Amer. Math. Soc. 79, 1–11 (1955).

Let K be an algebraic function field of genus g over a field of constants k. Using the method of algebraic geometry, in particular, the theory of jacobian varieties as developed by A. Weil, the author first gives a new proof of the well-known theorem that if $g>1$ and k is algebraically closed, K has only a finite number of automorphisms over k.

Without assuming that k is algebraically closed, the author then studies the structure of such a K with $g>1$ which has infinitely many automorphisms over k, and he obtains the following: 1) K is generated over k by elements $x, y_1, \ldots, y_n$ satisfying

$$y_i^p-y_i=a_ix^{p^{m_i}}+b_i \quad (i=1, \ldots, n),$$

where $p\neq0$ denotes the characteristic of K, m_i are positive integers and a_i, b_i are elements in k with a_i non-ϵ k^p; 2) the group of all automorphisms of K/k has a normal subgroup with a cyclic factor group of order prime to p, consisting of automorphisms of the form $\sigma(x)=x+\alpha$, $\sigma(y_i)=y_i+\beta_i$, where α, β_i are elements of k satisfying $\beta_i^p-\beta_i=a_i\alpha^{p^{m_i}}$. The author also remarks that for a suitable field k, we can really construct, in the way mentioned above, an algebraic function field K with genus $g>1$ which has infinitely many automorphisms over k. *K. Iwasawa* (Cambridge, Mass.).

Boseck, Helmut. Über Automorphismen algebraischer Funktionenkörper. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe 3, 361–362 (1954).

The author observes that the theorem to the effect that the group of automorphisms of a field of algebraic functions of one variable over an algebraically closed field is finite when the field is of genus >1 may be extended to the case where the basic field is perfect. Since then, M. Rosenlicht has given a complete classification of fields of functions of one variable with infinite automorphism groups [see the paper reviewed above]. *C. Chevalley.*

Differential Geometry

Parodi, Maurice. Cycloïde et spirale de Cornu. C. R. Acad. Sci. Paris 240, 1754–1755 (1955).

Making use of the Laplace transform the author establishes a one-to-one correspondence between the points of a cycloid and the spiral of Cornu. He shows that each arch of the cycloid is mapped continuously into a complete turn of the spiral. *H. P. Thielman* (Ames, Iowa).

Wunderlich, Walter. Über Loxodromen auf Zylindern 2. Grades. Monatsh. Math. **59**, 111–117 (1955).

A loxodrome is a curve which intersects the planes through a fixed axis a at a constant angle. The author studies those loxodromes l which lie on a quadratic cylinder Γ with the focal line a. They are the D-lines of Γ, i.e. their osculating spheres are tangent spheres of Γ. Such an l is also a loxodrome of the planes through each of the three other focal lines of Γ. A suitable dilatation in the direction of a transforms Γ and a into themselves and l into a helix on Γ whose angle with a is constant. The orthogonal projections of l onto the axial planes of Γ are determined and the limit cases that l is plane or Γ is parabolic are discussed.

P. Scherk (Sasakatoon, Sask.).

Löbell, Frank. Betrachtungen über Flächenabbildungen. X. "Ebenmässige" Abbildungen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1954**, 135–148 (1955).

[For parts I–IX see MR **9**, 614; **11**, 130, 131, 458.] Let the points of two surfaces of Euclidean 3-space, $\bar{x}(u, v)$ and $\bar{y}(\bar{u}, \bar{v})$ correspond if $\bar{u}=u$, $\bar{v}=v$. Consider the first fundamental forms of the two surfaces and also a third quadratic differential form, $d\bar{x}\cdot d\bar{y}$. Form $m^2=d\bar{y}^2/d\bar{x}^2$ and $n=d\bar{x}\cdot d\bar{y}/d\bar{x}^2$. The author proves that in general the three quadratic differential forms will be linearly dependent at a point if and only if m^2 and n take on their extreme values in precisely the same directions. Such a mapping is called "ebenmässig". This result is restated in several equivalent ways and the function $\Delta^2/(\bar{c}\cdot\vec{j}_1)^2(\bar{c}'\cdot\vec{j}_2)^2$ is suggested as a measure of how close a mapping $x\leftrightarrow y$ comes to being "ebenmässig". Here Δ is the determinant of the coefficients of the three quadratic forms, $\vec{j}_1=\bar{x}_u\times\bar{x}_v$, $\vec{j}_2=\bar{y}_u\times\bar{y}_v$, $\bar{c}$ and $\bar{c}'$ are the unit vectors of the positive normals of the surfaces $\bar{x}$, $\bar{y}$.

A. Schwartz (New York, N. Y.).

Löbell, Frank. Differentialformen in der Theorie der Flächenabbildungen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1954**, 149–157 (1955).

(See the paper reviewed above.) For the special case where the surfaces $\bar{x}$ and $\bar{y}$ have parallel normals there is a linear relationship between the three fundamental forms $d\bar{x}^2$, $d\bar{y}^2$, and $d\bar{x}\cdot d\bar{y}$ and a fourth one:

$$(*)\quad (\vec{j}_1\cdot\vec{j}_2)d\bar{x}^2-(\vec{j}_1\cdot\vec{j})d\bar{x}\cdot d\bar{y}+(\vec{j}_1\cdot\vec{j}_1)d\bar{y}^2+j\vec{j}_1\cdot d\bar{x}\times d\bar{y}=0.$$

Here $\vec{j}=\bar{x}_u\times\bar{y}_v-\bar{x}_v\times\bar{y}_u$, $j=\bar{x}_u\cdot\bar{y}_v-\bar{x}_v\cdot\bar{y}_u$. In this special case, the vector $d\bar{x}^2\vec{j}_2-d\bar{x}\cdot d\bar{y}\vec{j}+d\bar{y}^2\vec{j}_1+jd\bar{x}\times d\bar{y}$ vanishes. In general this vector determines a cone when du/dv varies, and this cone degenerates if and only if the mapping $d\bar{x}\rightarrow d\bar{y}$ has at least one fixed direction. The author also presents a generalization of (*), valid whether or not $\vec{j}_1$ is parallel to $\vec{j}_2$:

$$J_1\cdot J_2d\bar{x}^2-J_1\cdot Jd\bar{x}\cdot d\bar{y}+J\bar{c}\cdot d\bar{x}\times d\bar{y}+(\bar{c}\times d\bar{y})^2=0.$$

Here $J_1=\vec{j}_1/(\bar{c}\cdot\vec{j}_1)=\bar{c}$, $J_2=\vec{j}_2/(\bar{c}\cdot\vec{j}_1)$, $\vec{J}=\vec{j}/(\bar{c}\cdot\vec{j}_1)$, and $J=j/(\bar{c}\cdot\vec{j}_1)$.

A. Schwartz (New York, N. Y.).

Löbell, Frank. Dyaden in der Theorie der Flächenabbildungen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1954**, 335–345 (1955).

(See the papers reviewed above.) The affinity induced in each pair of corresponding tangent planes by the surface mapping $\bar{x}\rightarrow\bar{y}$ can be described conveniently by dyadics or affinors, $d\bar{y}=\mathfrak{A}d\bar{x}$. First, starting with the case of parallel normals for surfaces $\bar{x}$ and $\bar{y}$, the author shows that the affinor $\mathfrak{A}$ satisfies $\bar{\mathfrak{A}}\mathfrak{A}-2H\mathfrak{A}+K\mathfrak{I}-J\mathfrak{A}^*=0$. Here $\bar{\mathfrak{A}}$ is the conjugate of $\mathfrak{A}$, $\mathfrak{A}^*=\bar{c}\times\mathfrak{A}$, $2H=(\vec{j}_1\cdot\vec{j})/(\vec{j}_1\cdot\vec{j}_1)$, $K=(\vec{j}_1\cdot\vec{j}_2)/(\vec{j}_1\cdot\vec{j}_1)$, and $\mathfrak{I}$ is the identity affinor. Second, if corresponding normals are not parallel, we find

$$(\dagger)\quad \bar{\mathfrak{A}}\mathfrak{A}-2H\mathfrak{A}-J\mathfrak{A}^*+K\mathfrak{I}=\vec{J}\cdot\vec{J}-2H\vec{J}\cdot J_1+KJ_1\cdot J_1.$$

Let $\vec{f}_1$ and $\vec{f}_2$ be the unit vectors in the principal directions on surface $\bar{x}$ and $\vec{f}_1'$, $\vec{f}_2'$ the corresponding directions on $\bar{y}$. Then the affinor $\mathfrak{A}=\vec{f}_1'\vec{f}_1+\vec{f}_2'\vec{f}_2$ can be used. If one prefers an affinor of rank 3, one can use $\mathfrak{B}=\mathfrak{A}-2H\bar{c}\bar{c}$. $\mathfrak{B}$ also satisfies (†). Third, the author works out the Cayley-Hamilton equation for $\mathfrak{B}$: $\mathfrak{B}^3+(K-4H^2)\mathfrak{B}+2KH\mathfrak{I}=0$.

A. Schwartz (New York, N. Y.).

Čahtauri, A. I. Applications of the intrinsic geometries of plane nets in the theory of surfaces. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **20**, 89–130 (1954). (Russian)

This paper is a continuation of the author's previous work on planar nets associated with a surface. It is an elaboration of classical results dealing with nets of Lie, Laplace, Fubini etc. The last part deals with deformations preserving various nets.

M. S. Knebelman (Pullman, Wash.).

Rinow, W. Über eine axiomatische Begründung der inneren Geometrie der Flächen. Acta Math. Acad. Sci. Hungar. **5**, supplementum, 145–152 (1954). (Russian summary)

This is the beginning of an investigation which looks very promising. Its aim is a configurational characterization of the plane curve systems which occur as extremals in symmetric two-dimensional variational problems. At each point the tangents of the solutions form a pencil of lines, in which cross-ratios can be defined in a natural way. This paper assumes that a system of curves is given in the plane which satisfies the axioms of order and connection, and also continuity. It then gives, without proof, a limit form of Desargues' Theorem which guarantees that an invariant cross-ratio can be defined for the curves through a point. This is not yet sufficient for the existence of a coordinate system in terms of which the given curves are of class C^1, but the author states that he knows a dual form of the above limit theorem which will produce this effect, whereas he has not yet arrived at conclusive results regarding the question when the curves are of class C^2. [For earlier expositions of these results see MR **15**, 466.]

H. Busemann.

Grotemeyer, Karl-Peter. Integralsätze bei infinitesimalen Verbiegungen von geschlossenen Raumkurven. Arch. Math. **6**, 250–252 (1955).

An infinitesimal deformation $\mathfrak{x}(s)+\epsilon\delta\mathfrak{x}(s)$ of a closed curve $\mathfrak{x}(s)$ in E^3, where s is arc length, is called rigid if $\dot{\mathfrak{x}}\cdot\delta\dot{\mathfrak{x}}=0$. If $\mathfrak{n}$, $\mathfrak{b}$, κ, τ are respectively the principal normal, binormal, curvature, torsion of $\mathfrak{x}(s)$, then

$$\oint\delta\kappa ds=-\oint\tau(\mathfrak{b}\cdot\delta\dot{\mathfrak{x}})ds,\quad \oint\delta\tau ds=+\oint\kappa(\mathfrak{b}\cdot d\dot{\mathfrak{x}})ds,$$

$$\oint\delta\kappa\mathfrak{x}ds=-\oint\tau\{\delta\mathfrak{x}\times\mathfrak{n}+\mathfrak{x}(\mathfrak{b}\cdot\delta\dot{\mathfrak{x}})\}ds.$$

H. Busemann (Los Angeles, Calif.).

Hellwig, Günter. Über die Verbiegbarkeit von Flächenstücken mit positiver Gauszscher Krümmung. Arch. Math. **6**, 243–249 (1955).

Under strong differentiability hypotheses it is shown that a piece S of a convex surface homeomorphic to a disk, whose integral curvature is less than 4π, is not rigid. Much stronger

results are found in Pogorelov's "Deformation of curved surfaces" [Gostehizdat, Moscow-Leningrad, 1951; MR **14**, 400, 1278] and later results of Pogorelov together with A. D. Alexandrov's gluing theorem allow us to eliminate all differentiability hypotheses. It is also shown that, if the lines of curvature on S are isothermal, then an infinitesimal rigid deformation of S leaving the mean curvature H on the boundary fixed, leave H everywhere fixed.

H. Busemann (Los Angeles, Calif.).

Efimov, N. V. Some theorems on surfaces of negative curvature. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 101–105 (1955). (Russian)

On a surface of class C^3 in E^3 with negative Gauss curvature K consider a simply connected domain G which may, however, cover parts of S several times. From a point p in G lay off on each geodesic in G an arc of length r, thus obtaining a generalized geodesic circle C_r. The following is proved: if $K\leqq -1$, then there is a universal estimate $r\leqq f(\mu)$, for instance $f(\mu)=\pi(1+\mu^2)^{1/2}$, where μ is the maximum of the absolute value of the normal curvatures of S in C_r. The following is a corollary: On a complete surface of class C^3 in E^3 either the sup of the absolute value of the mean curvature is ∞, or the sup of the Gauss curvature is non-negative. The corollary is essentially equivalent to a theorem stated without proof by J. J. Stoker [Bull. Amer. Math. Soc. **60**, 258 (1954)].

H. Busemann.

Godeaux, Lucien. Sur les congruences W dont une des nappes focales est une surface ayant ses asymptotiques des deux familles dans des complexes linéaires. Bull. Soc. Roy. Sci. Liège **24**, 79–89 (1955).

Les congruences W dont l'une des nappes focales a ses asymptotiques des deux familles dans des complexes linéaires, ont fait l'objet d'une étude antérieure de M. Rozet et de Mme. Legrain-Pissard [Bull. Soc. Roy. Sci. Liège **23**, 280–296 (1954); MR **16**, 513]. L'auteur reprend ici la question en montrant comment on peut la rattacher aux méthodes qu'à diverses reprises il a appliquées à l'étude des congruences W, et qui mettent en jeu les propriétés (opportunément approfondies) des suites de Laplace auxquelles appartiennent les points représentatifs, sur l'hyperquadrique de Klein de E_5, des tangentes asymptotiques des surfaces de E_3.

P. Vincensini (Marseille).

Mishra, R. S. On the congruences of curves in a subspace. J. Math., Tokyo **1**, 63–66 (1953).

Dans un article antérieur [Ram Behari et R. S. Mishra, Proc. Nat. Inst. Sci. India **15**, 85–92 (1949); MR **11**, 53], l'auteur a donné une expression, dans l'espace euclidien à trois dimensions, de la quantité $v_i\equiv X^\alpha\lambda^\alpha{}_{;j}$, où X^α et λ^α sont, respectivement, les cosinus directeurs de la normale au point courant x^α d'une surface quelconque, et du rayon d'une congruence rectiligne quelconque issu de x^α. Il a, par la suite, étendu les expressions obtenues aux congruences de courbes d'un espace Riemannien V_{n+1}, telles que par chaque point d'une hypersurface V_n de cet espace il passe une courbe et une seule de la congruence.

Dans l'article actuel il donne les expressions analogues aux précédentes pour un système de $m-n$ congruences courbes d'un espace Riemannien à m dimensions, telles que par chaque point d'une variété V_n de V_m il passe une courbe et une seule de chaque congruence. Il envisage, en particulier, le cas où V_n est une hypersurface V_{m-1}, et celui où, dans cette même hypothèse, les courbes de la congruence considérée sont orthogonales à l'hypersurface.

P. Vincensini (Marseille).

Geĭdel'man, R. M. On the theory of pseudo-congruences and congruences of planes of a multidimensional hyperbolic space and of congruences of spheres of a multidimensional conformal space. Mat. Sb. N.S. **36(78)**, 209–232 (1955). (Russian)

As is well known, the conformal geometry of a n-dimensional euclidean space M_n is equivalent to the projective metric geometry of a hyperbolic space S'_{n+1} leaving invariant a hyperquadric Q_n with equation

$$A^2\equiv\lambda_0^2+\lambda_1^2+\ldots+\lambda_n^2-\lambda^2_{n+1}=0,$$

where $\lambda_0,\ldots,\lambda_{n+1}$ are homogeneous coordinates. For this reason the first part of the paper is concerned with k-parameter pseudo-congruences of $(k-1)$-dimensional planes which do not intersect Q_n and the last part translates these results into pseudo-congruences of spheres and planes in M_n. The method consists of selecting a frame of reference A_0, $A_1,\ldots,A_{n+1}$ normalized with reference to the hyperquadric and studying the Pfaffian forms $\omega_\beta{}^\alpha$ which define a displacement of this reference frame. The analysis is based on the recent work of the Russian school of differential geometers.

M. S. Knebelman (Pullman, Wash.).

Mokriščev, K. K. On some classes of curves of Lobačevskiĭ space. Dokl. Akad. Nauk SSSR (N.S.) **100**, 9–11 (1955). (Russian)

In three-dimensional hyperbolic space with space constant σ^{-1}, let normals of the curve C which form the constant angle ω with the principal normals of C be binormals of another curve C'. Denote the curvatures of C, C' by κ, κ', the torsions by τ, τ'. The distance a between corresponding points of C and C' is constant. The natural equation of C has the form $\kappa^2\cos^2\omega+\tau^2-2\sigma\kappa\coth 2a\sigma\cdot\cos\omega+\sigma^2=0$. This implies that $\omega=\pi/2$ cannot exist for real C, C'. If $\omega=0$ then the natural equation of C' is $\tau'=\sigma\cot a\sigma\cdot\tan\int\kappa' ds'$ and $\tau\tau'-[\sigma\sin\varphi\cdot\operatorname{csch} a\sigma]^2$, where φ is the angle between corresponding normal planes of C and C'. There are also some new relations for pairs of curves with common principal normals.

H. Busemann (Los Angeles, Calif.).

Petkantschin, B. Regelscharen isotroper Geraden im elliptischen Raum. Bŭlgar. Akad. Nauk. Izv. Mat. Inst. **1**, no. 2, 171–198 (1954). (Bulgarian. Russian and German summaries)

Let J be the absolute locus of the three-dimensional complex elliptic space N_3. Let the line $x(u)p(u)$ be a proper tangent of J at $p(u)$, where $x(u)$ and $p(u)$ are analytic functions of u. These lines generate a ruled surface F with isotropic rulings. A projective coordinate system with reference points a_0, a_1, a_2, a_3, f in N_3 is called cyclic, if a_0, a_2 are finite orthogonal points on the polar of a_1a_3 and the polars of f with respect to J and the tetrahedron $a_0a_1a_2a_3$ coincide. It is shown that with every line $x(u)p(u)$ a cyclic coordinate system can be associated in an invariant way. The Frenet formulae with respect to this moving coordinate system are derived. In analogy to a result of Duschek on ruled surfaces in E_3 it is shown that the central curve of F is the only line of curvature different from the isotropic lines.

H. Busemann (Los Angeles, Calif.).

Mayer, O. Familles R de courbes sur les surfaces réglées de l'espace euclidien. Acad. Repub. Pop. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Ser. I. **5**, no. 3–4, 13–47 (1954). (Romanian. Russian and French summaries)

L'auteur étudie, du point de vue métrique, les familles (simples) de courbes régulières d'une surface réglée quelconque, déterminant des divisions homographiques sur les différentes génératrices rectilignes de la surface. Si v est le paramètre fixant les différentes génératrices et u le paramètre projectif variable sur chacune d'elles, les familles jouissant de la propriété indiquée (familles R) sont définies par une équation de Riccati

$$\frac{du}{dv}+A(v)u^2+2B(v)u+C(v)=0.$$

Portant plus spécialement l'attention sur les familles non paraboliques ($A\neq0$), l'auteur envisage les transformations des familles précédentes qui conservent les distances sur les génératrices rectilignes. Il met en evidence, pour ces transformations, l'existence de deux invariants, auxquels se rattache la considération d'une courbe (ligne de striction) liée de façon covariante à la famille envisagée. Si l'on introduit, à la place de u, le paramètre angulaire φ définissant le plan tangent en chaque point d'une génératrice, l'équation de Riccati ci-dessus est remplacée par une autre équation, qui conduit elle aussi à la considération de deux invariants et d'un couple de lignes de striction. L'auteur étudie ensuite quelques particularisations intéressantes de familles R, les courbes de la famille étant soit des asymptotiques virtuelles, soit des lignes de courbure, isogonales, de Tchebycheff ou géodésiques, en déterminant, dans chaque cas, les surfaces R supports des familles R envisagées.

P. Vincensini (Marseille).

44 ***Bol, Gerrit. Projektive Differentialgeometrie. 2. Teil.** Vandenhoeck & Ruprecht, Göttingen, 1954. 372 pp. DM 27.50.

As a continuation of the first volume with the same title by the author [1950; MR **11**, 539], this book, beginning with Chapter V, contains four chapters, all of which are devoted to the theory of surfaces in three-dimensional projective space.

The first part of Chapter V treats exhaustively the half-invariant differentiation, which was initiated in a previous paper [Comment. Math. Helv. **18**, 129–153 (1946); MR **7**, 327] by the author and is used as a fundamental tool throughout this volume. By means of this differentiation, the author then sets up a system of linear homogeneous partial differential equations defining a proper surface, referred to its asymptotic curves, in a three-dimensional projective space. Moreover, there are obtained local plane, as well as point, coordinate systems, the quadric of Lie, the quadrics of Darboux, the canonical pencils of lines associated with a point of the surface and power series expansions for the surface, together with existence and uniqueness theorems in the projective theory of surfaces.

In Chapter VI, the author gives a simple geometric interpretation to the projective area element of a surface; and geometrically introduces the quadric of Wilczynski, the edges of Green, the directrices of Wilczynski and other covariant quadrics and canonical lines associated with a point of a surface. Some relations between the projective and the affine theories of surfaces are also included. For instance, an affine minimal surface is characterized by the property that all its quadrics of Lie are paraboloids and an affine sphere by the property that its directrices of Wilczynski of the first kind coincide with its affine normals.

Chapter VII deals with curves and systems of curves on a surface. It contains, among other things, the asymptotic osculating quadrics of Bompiani, the quadric of Moutard, the transformation of Bell, hypergeodesics with their geometric construction, the projective geodesics of Fubini, pangeodesics, a projective analogue of the well-known Gauss-Bonnet formula and a simplified representation of Fubini's theory of line congruences including W- and R-congruences.

In the last chapter, Wilczynski's system of partial differential equations of a surface, referred to its asymptotic curves, is used to study the envelopes of the quadrics of Lie, the figure of Demoulin and the sequence of Godeaux.

Asymptotic parameters are used throughout this volume except one section in Chapter VII, where conjugate nets are briefly discussed. As in the first volume almost every section is followed by an extensive list of problems and supplementary theorems, which make this volume cover its material well. There is no bibliography at the end of this volume, all new references being placed in the text.

C. C. Hsiung (Bethlehem, Pa.).

Villa, Mario. Problemi integrali sulle trasformazioni puntuali. Compositio Math. **12**, 137–146 (1954).

L'A. studia alcune questioni sulle trasformazioni puntuali fra due piani $\pi(x, y)$, $\bar\pi(\bar x, \bar y)$, col metodo del riferimento mobile di E. Cartan, e le mette in relazione con risultati precedenti ottenuti da lui ed altri, in particolare da E. Čech. Più precisamente, siano T_1, T_2 due trasformazioni puntuali fra π, $\bar\pi$, che si approssimino fino all'intorno di ordine $s>0$ di una coppia di punti regolari corrispondenti, O, $\bar O$, e sia p una retta di π per O, al cui E_{s+1} (inflessionale) per O corrisponderanno in T_1, T_2 due E_{s+1} per $\bar O$; i fasci che proiettano da un punto S di $\bar\pi$ questi due E_{s+1} hanno un contatto analitico d'ordine almeno $s+1$ se e solamente se S appartiene a una retta $\bar p$ per $\bar O$: alla $\bar p$, d'altro canto, corrisponde una retta p' nella proiettività fra i fasci di centri $\bar O$, O, subordinata da entrambe le trasformazioni T_1, T_2; la corrispondenza γ: $p\rightarrow p'$ è detta corrispondenza linearizzante d'ordine $s+1$ relativa a T_1, T_2. Se T_2 è un'omografia tangente a T_1, K, si ha la corrispondenza K-linearizzante di Čech [Časopis Pěst. Mat. Fys. **74**, 32–48; **75**, 123–136, 137–158 (1950); MR **12**, 534; **13**, 158]. Applicando il metodo del riferimento mobile, e introducendo quindi due opportuni pfaffiani ω_1, ω_2, è possibile definire due forme quadratiche nelle ω, Ω_1, Ω_2, associate a una corrispondenza K-linearizzante; scelto poi opportunamente il riferimento, si definiscono—a partire dalle Ω—due forme cubiche negli stessi argomenti, che dànno luogo a una trasformazione linearizzante del 3° ordine.

V. Dalla Volta (Roma).

Arghiriade, E. Compléments à la théorie des quadriques osculatrices d'une surface. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 573–577 (1954). (Romanian. Russian and French summaries)

Let x be a point of a surface S in projective 3-space; if Q is a quadric having a 2nd-order contact with S at x, it cuts S, as well known, in a curve C having at x a triple point: denote by T the three tangents at C at x. There is a unique line $\mathbf{r}$, defined by the condition that its first polar with respect to T is composed with two conjugate tangents with respect to S; according to Mayer [Ann. Sci. Univ. Jassy **21**, 1–77 (1935)] $\mathbf{r}$ is the principal line with respect

to the tangents T at x. The author proves first that the osculating quadrics considered by Davis and Green [Green, Amer. J. Math. 60, 649–666 (1938)] are the same studied later by I. Popa [Acad. Roum. Bull. Sect. Sci. 18, 113–116 (1937)], and by B. Gambier [J. Math. Pure Appl. (9) 15, 151–162 (1936)]. In the following paragraphs the author considers also the osculating quadrics D_k studied by P. O. Bell [Duke Math. J. 5, 784–788 (1939); MR 1, 84] and proves that they may be defined as the osculating quadrics Q for which the tangents T have as principal line the 2nd canonical line. Finally the quadrics D_k are studied from the tangential point of view.

V. Dalla Volta (Rome).

Haantjes, J. On X_m-forming sets of eigenvectors. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17, 158–162 (1955).

A. Nijenhuis [same Proc. 54, 200–212 (1951); MR 13, 281] has studied the question whether the covariant eigenvectors of a tensor $T_\lambda^{\cdot\kappa}$ in an n-dimensional manifold X_n with n distinct eigenvalues are X_{n-1}-forming. He proved that the covariant eigenvectors are X_{n-1}-forming if and only if the tensor

$$H_{\mu\lambda}^{\cdot\cdot\kappa}=2T_{[\mu}^{\cdot\nu}\partial_{|\nu|}T_{\lambda]}^{\cdot\kappa}-2T_\nu^{\cdot\kappa}\partial_{[\mu}T_{\lambda]}^{\cdot\nu}$$

can be written in the form

$$H_{\mu\lambda}^{\cdot\cdot\kappa}=\overset{0}{a}_{[\mu}A_{\lambda]}^{\kappa}+\overset{1}{a}_{[\mu}T_{\lambda]}^{\cdot\kappa}+\overset{2}{a}_{[\mu}\overset{2}{T}_{\lambda]}^{\cdot\kappa}+\dots+\overset{n-2}{a}_{[\mu}\overset{n-2}{T}_{\lambda]}^{\cdot\kappa},$$

where $\overset{2}{T}_\lambda^{\cdot\kappa}=T_\mu^{\cdot\kappa}T_\lambda^{\cdot\mu}$, etc. This condition has the disadvantage that it contains some unknown quantities. The author attacks the same problem in a different way and proves the following theorem.

Let $T_\lambda^{\cdot\kappa}$ be a tensor such that to each root with multiplicity k of the characteristic equation belongs a set of r linearly independent covariant eigenvectors. Then the E_{n-r} $(=(n-r)$-dimensional linear space) determined by these vectors are X_{n-r}-forming if and only if

$$H_{\nu\sigma}^{\cdot\cdot\kappa}T_\mu^{\cdot\nu}T_\lambda^{\cdot\sigma}+2H_{\nu[\lambda}^{\cdot\cdot\sigma}T_{\mu]}^{\cdot\nu}T_\sigma^{\cdot\kappa}+H_{\mu\lambda}^{\cdot\cdot\nu}T_\sigma^{\cdot\kappa}T_\nu^{\cdot\sigma}=0.$$

Here the condition is given in a form which does not contain unknown quantities. The author applies this theorem to a symmetric tensor field $h_{\mu\lambda}$ in an n-dimensional Riemannian space V_n and gets the following theorem: The principal m-directions of a real symmetric tensor field $h_{\mu\lambda}$ in an ordinary V_n are V_{n-m}-normal if and only if the tensor

$$H_{\mu\lambda}^{\cdot\cdot\kappa}=2h_{[\mu}^{\cdot\nu}\nabla_{|\nu|}h_{\lambda]}^{\cdot\kappa}-2h_\nu^{\cdot\kappa}\nabla_{[\mu}h_{\lambda]}^{\cdot\nu}$$

satisfies the condition

$$H_{\nu\sigma}^{\cdot\cdot\kappa}h_\mu^{\cdot\nu}h_\lambda^{\cdot\sigma}+2H_{\nu[\mu}^{\cdot\cdot\sigma}h_{\lambda]}^{\cdot\nu}h_\sigma^{\cdot\kappa}+H_{\mu\lambda}^{\cdot\cdot\nu}h_\nu^{\cdot\sigma}h_\sigma^{\cdot\kappa}=0.$$

K. Yano (Tokyo).

Haimovici, Adolf. Sur quelques invariants attachés à un couple de vecteurs d'un espace à connexion affine à deux dimensions. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6, 31–48 (1954). (Romanian. Russian and French summaries)

The A_2 under consideration admit, under parallel transformation, an invariant belonging to any couple of vectors X^i, Y^i attached to a point. The classification depends on the rank of the two matrices

$$R=\begin{Vmatrix} R^1{}_{112} & R^1{}_{212} & R^2{}_{112} & R^2{}_{212} \\ R^1{}_{112,1} & R^1{}_{212,1} & R^2{}_{112,1} & R^2{}_{212,1} \\ R^1{}_{112,2} & R^1{}_{212,2} & R^2{}_{112,2} & R^2{}_{212,2} \end{Vmatrix}$$

where the commas indicate covariant differentiation, and

$$A=\begin{Vmatrix} a_1{}^1 & a_1{}^2 & a_2{}^1 & a_2{}^2 \\ b_1{}^1 & b_1{}^2 & b_2{}^1 & b_2{}^2 \\ c_1{}^1 & c_1{}^2 & c_2{}^1 & c_2{}^2 \\ d_1{}^1 & d_1{}^2 & d_2{}^1 & d_2{}^2 \\ e_1{}^1 & e_1{}^2 & e_2{}^1 & e_2{}^2 \\ \vdots & \vdots & \vdots & \vdots \\ a^1{}_{1,rs} & a^2{}_{1,rs} & a^1{}_{2,rs} & a^2{}_{2,rs} \end{Vmatrix}, \qquad \begin{aligned} a_i{}^j&=R^j{}_{i12} \\ b_i{}^j&=R^j{}_{i12,1} \\ c_i{}^j&=a_i{}^mb_m{}^j-a_m{}^jb_i{}^m \\ d_i{}^j&=a_i{}^mc_m{}^j-a_m{}^jc_i{}^m \\ e_i{}^j&=b_i{}^mc_m{}^j-b_m{}^jc_i{}^m \\ &\vdots \\ a^j{}_{i,rs}&=R^j{}_{i12,rs}. \end{aligned}$$

The invariant $X^1Y^2-X^2Y^1$ appears when the rank of R and A is three, except when a certain relation between the R_{ij} exists. The cases in which R and A have rank <3 are also analyzed.

D. J. Struik (Cambridge, Mass.).

Kobayashi, Shôshichi. Une remarque sur la connexion affine symétrique. Proc. Japan Acad. 31, 14–15 (1955).

The following theorem is proved: Let $V=G/H$ be a compact symmetric homogeneous space, such that H has a finite number of connected components. Then there exists on V a symmetric Riemannian connection, which is invariant by a compact subgroup K of G, K acting transitively on V.

S. Chern (Chicago, Ill.).

Vrănceanu, G. Propriétés différentielles globales des espaces A_n à groupe maximum G_{n^2}. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6, 49–59 (1954). (Romanian. Russian and French summaries)

The affine spaces A_n with an invariant Pfaffian and a maximum group G_{n^2} of automorphisms can be characterized by coefficients

$$\Gamma^1{}_{11}=\mu,\quad \Gamma^h{}_{k1}=\rho\delta_k{}^h,\quad \Gamma^h{}_{1k}=0,\quad \Gamma^h{}_{11}=(\lambda+\mu\rho)x^h,\quad \Gamma^i{}_{hk}=0$$
$$(h, k=2, \dots, n;\ \mu, \rho, \lambda \text{ constants}).$$

This has been shown in the author's "Lectures on differential geometry," v. II, [Acad. Repub. Pop. Române, 1951, p. 66; MR 16, 1049]. In the present paper we find expressions for the auto-parallel curves of these A_n for the case without torsion $(\rho=0)$; there are three cases depending on the character of the roots of the quadratic equation $r^2+\mu r-\lambda=0$. It is also shown, by a suitable transformation, that A_n can be given constant connection coefficients. The structure of the G_{n^2} is analyzed, and it is demonstrated that there exist two symmetrical spaces in the sense of Cartan, given by

$$\Gamma^1{}_{11}=\Gamma^h{}_{ki}=\Gamma^h{}_{ik}=\Gamma^i{}_{hk}=0,\quad \Gamma^h{}_{11}=\pm x^h$$
$$(h, k=2, \dots, n;\ i=1, \dots, n).$$

D. J. Struik (Cambridge, Mass.).

Kručkovič, G. I. Invariant criteria of spaces V_3 with the group of motions G_4. Uspehi Mat. Nauk (N.S.) 10, 1(63), 129–136 (1955). (Russian)

The main interest of this paper is due to the fact that for Riemann spaces V_n with a maximal group of motions $G_{\frac{1}{2}n(n-1)+1}$, V_3 with a group G_4 plays an exceptional role. In a sense this paper is a continuation of classification of V_3's with groups G_2, G_3, G_4 [Uspehi Mat. Nauk (N.S.) 9, no. 1(59), 3–40 (1954); MR 15, 986], V_3's with G_1 and G_6 being classical. The author transcribes the Killing's equations and their integrability conditions in terms of the coefficients of rotation and the principal directions of Ricci. The problem is then separated into two possible cases. In the first case the V_3 with a G_4 admits an orthogonal trihedral of principal directions. In this case the characteristic roots are necessarily $\rho_1=\rho_2\neq\rho_3$ and the main result is that for such a V_3 the roots must be constant and the coefficients of rotation must satisfy $\gamma_{231}+\gamma_{312}=0$, $\gamma_{311}=\gamma_{322}=\gamma_{312}=\gamma_{323}=0$.

The author also gives a more geometric characterization of these spaces. In the second case V_3 does not admit an orthogonal trihedral of principal directions; in this case the metric can be one of four types as the author has shown (loc. cit.). In the first three of these spaces $\rho_1=\rho_2=\rho_3=0$ and in the fourth $\rho_1=\rho_2=\rho_3=-\frac{1}{2}e_1$. If the space is conformally euclidean it is necessary and sufficient that $R_{ij}=Kf_{.i}f_{.j}$; $f_{.ij}=Lf_{.i}f_{.j}$, where K, L are constants $K\neq 0$. If V_3 is not conformally euclidean, then $R_{ij}+\rho g_{ij}=e_1\mu_i\mu_j$, $\rho\neq 0$ and $\mu_{i,j}+\mu_{j,i}=0$, where μ_i is a multiple of a gradient.

M. S. Knebelman (Pullman, Wash.).

✶**Debever, R. Le groupe d'holonomie des variétés de groupe de Lie, à connexion conforme normale.** III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 56–58. Fédération belge des Sociétés Scientifiques, Bruxelles.

The author shows that with a compact semi-simple Lie group there is naturally associated a space with a conformal connexion. He then proves that if the order of the group exceeds three, then the holonomy group of this space is the group of conformal transformations which leave invariant a fixed imaginary hypersphere. If the order of the group is precisely three, then the corresponding holonomy group is discrete. In effect these results were announced by E. Cartan [J. Math. Pures Appl. (9) 6, 1–119 (1927), p. 119], who indicated a method of proof. The author's proof, based on the introduction of polyspherical frames naturally associated with the conformal group, is simpler than Cartan's.

T. J. Willmore (Liverpool).

Papy, Georges. Sur la définition intrinsèque des vecteurs tangents. C. R. Acad. Sci. Paris **241**, 19–20 (1955).

An intrinsic definition of the tangent vectors of an analytic manifold was given by Chevalley [Theory of Lie groups, Princeton, 1946, p. 76; MR **7**, 412]. This definition can be used to give the tangent vectors of a differentiable manifold in the ordinary sense only if the manifold is of class C^r, $r>1$. The author shows that, if the manifold is of class C^1, the tangent vectors at a point in this sense form an infinite-dimensional vector space.

Reviewer's note. In order to adapt the definition to manifolds of class C^1 to give the tangent vectors in the ordinary sense, it is necessary to introduce the differentiation of a class of functions. Two functions f, g, both defined in a neighborhood of a point p (which may vary with the function), are said to be equivalent or to belong to the same class if the gradient of $f-g$ vanishes at p. A tangent vector can then be defined as a mapping of a class of functions to the field of real numbers, with the same conditions satisfied.

S. Chern (Chicago, Ill.).

Ganea, Tudor. On the Prüfer manifold and a problem of Alexandroff and Hopf. Acta Sci. Math. Szeged **15**, 231–235 (1954).

A proof is given of the non-normality of the Prüfer manifold. This has been shown independently by Calabi and Rosenlicht [Proc. Amer. Math. Soc. **4**, 335–340 (1953); MR **15**, 351].

W. Kaplan (Ann Arbor, Mich.).

NUMERICAL AND GRAPHICAL METHODS

✶**Hrenov, L. S. Pyatiznačnye tablicy trigonometričeskih funkcii s argumentom, vyražennym v časovoĭ mere. [Five-place tables of the trigonometric functions with argument expressed in hourly measure.] 2d ed.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 172 pp. 7.70 rubles.

The main table gives values of the six trigonometrical functions at interval of 4 seconds of time (i.e. 1 minute of arc), usually to 5S, in the range 0 to 3 hours. (Whenever the leading figure is unity, five further digits are given.) First differences are usually given, with their proportional parts alongside the tables. An auxiliary table, new in this second edition, gives cot x, cosec x to 5S for $x=0(.1^s)08^m(1^s)40^m$. There is a table of $\sin^2(\frac{1}{2}x)$ for $x=0(4^s)12^h$. Entries with terminal 5 are marked with $\pm$ to indicate in what direction rounding should be made. There is a collection of conversion tables, constants and formulae from plane and spherical trigonometry and there are worked examples showing the direct and inverse use of the tables. The tables are clearly printed.

There are no references to sources, nor is there a description of the construction. It is stated that the entries are correct to half a unit in last place. The tables should be convenient for those who require something between the two volumes issued by L. J. Comrie [Four-figure tables . . . argument in time, London, 1931] which gives 4+ decimals at 10^s interval, and the British Nautical Almanac Office [Seven-figure trigonometrical tables for every second of time, H. M. Stationery Office, London, 1939], which gives values to 7D at 1^s interval.

John Todd.

✶**Tablicy e^x i e^{-x}. [Tables of e^x and e^{-x}.]** Izdat. Akad. Nauk SSSR, Moscow, 1955. 145 pp. 16.60 rubles.

The main table gives $e^{\pm x}$ to 10S for $x=0(.001)10$. There follows a 20D table for $x=m(10^{-n})$ for $n=3$, 6, 9 and $m=1(1)999$ and a 20S table for $x=1(1)100$. The main tables were computed by repeated multiplication by $e^{.001}$ with checks from the power series evaluated at $x=0.5(5)10$. The tables for $n=6$, 9, $m=1(1)999$ were computed from the power series and that for $n=3$, $m=1(1)999$ by interpolation, using the Gaussian formula. The table for $x=1(1)100$ was calculated by repeated multiplication by e, with independent checks at $n=20(20)100$. All tables were checked by differencing. It is stated that the values given are correct to within .6 units in the last place. There are worked examples showing the use of the tables for arguments of various size and precision. The table is clearly printed. *John Todd.*

✶**Sibagaki, W. 0.01% tables of modified Bessel functions, with the account of the methods used in the calculation.** Baifukan, Tokyo, 1955. iv+xxi+130 pp. 1050 yen. (Title pages in Japanese and English)

This handy book tabulates the modified Bessel functions $x^{\frac{1}{2}}I_n(x)$ and $x^{\frac{1}{2}}K_n(x)$, $n=0(1)22$ for I_n and 23 for K_n, $x=0(.01)1(.02)5(.04)25$, to 4 or 5 significant figures, with differences corresponding to the increment $\Delta x=0.01$. This table is an improved version of an earlier mimeographed 1945 edition [see Math. Tables Aids Comput. 5, 80 (1951)] and differs from its predecessor in tabulating $x^{\frac{1}{2}}I_n(x)$ and $x^{\frac{1}{2}}K_n(x)$ instead of $I_n(x)$ and $K_n(x)$ for $x\leq 5$. This new edition contains much more introductory material: a section on the description, use and, preparation of the tables is in both

English and Japanese; the later sections, in Japanese only, deal with the following topics: properties of the modified Bessel functions, derivation of Debye's asymptotic formulae by steepest descent, and a description of a method of numerical integration of ordinary differential equations of the form $y''=f(x,y)$ with two long examples illustrating the special case of $f(x,y)=(1+x^{-2}(n^2-\frac{1}{4}))y$. Graphs of both $x^{\frac{1}{2}}I_n(x)$ and $x^{\frac{1}{2}}K_n(x)$ are provided. A short critical table of Bessel's second-order coefficient at the end aids in quadratic interpolation, sufficient nearly everywhere. The printing, paper, and hard-cover binding also present great improvements over the earlier version. There is a short list of errata on an insert. *H. E. Salzer* (Washington, D. C.).

Miyagawa, Matsuo. Roots x_n of

$$J_0(x)Y_1(kx)-Y_0(x)J_1(ks)=0.$$

Mem. Fac. Tech. Tokyo Metro. Univ. **1955**, no. 5, 313–314 (1955).

The author gives 3D values of the first five roots of the equation mentioned in the title for $k=1.5$, 2, 3, 4, 5.

A. Erdélyi (Pasadena, Calif.).

***Dijkstra, E. W., and van Wijngaarden, A. Table of Everett's interpolation coefficients.** Computation Department of the Mathematical Centre, Amsterdam, Rep. R294, 204 pp. (1955). fl. 6.

These coefficients are for use in the interpolation formula

$$f(p)=\{(1-p)f(0)+pf(1)\}\\ +\sum_{n=1}\{E_0^{2n}(p)\delta^{2n}f(0)+E_1^{2n}(p)\delta^{2n}f(1)\}.$$

Here $E_r^{2n}(p)=(-1)^{r+1}\binom{p+n+r-1}{2n+1}$ for $r=0, 1$. The table, which gives these coefficients to 7D for $p=0(.0001)1$ for $r=0$, 1 and $n=1$, 2, 3, was reproduced from copy prepared by the relay computer ARRA of the Math. Centrum. It will be more convenient in some practical cases than either the tables of A. J. Thompson [Table of the coefficients of Everett's central-difference interpolation formula, 2nd ed., Cambridge, 1943; MR **4**, 202], or the National Bureau of Standards' Tables of Lagrangian interpolation coefficients [Columbia Univ. Press, 1944; MR **5**, 244] for these tables are, mainly, at interval .001, and have been condensed by using the symmetry property $E_0^{2n}(p)=E_1^{2n}(1-p)$ so that interpolation and entry at two places is often necessary.

John Todd (Washington, D. C.).

Uhler, Horace S. Errata: Hamartiexéresis as applied to tables involving logarithms. Proc. Nat. Acad. Sci. U. S. A. **41**, 183 (1955).

See same Proc. **40**, 728–731 (1954); MR **16**, 175.

David, F. N., and Kendall, M. G. Tables of symmetric functions. V. Biometrika **42**, 223–242 (1955).

This is the fifth and final pair of tables giving the relations connecting the five sorts of symmetric functions considered previously by the authors [the four previous tables appeared in Biometrika **36**, 431–449 (1949); **38**, 435–462 (1951); **40**, 427–446 (1953); MR **11**, 488; **13**, 781; **15**, 471]. In the notation of these reviews the present tables are designated by US-SU, that is they give the relations between the unitary and the power sum symmetric functions up to and including functions of weight 12. For example, the coefficients in the following two identities are read directly from the table of weight 4.

$$s_3s_1=a_1^4-3a_2a_1^2+3a_3a_1,$$
$$4!a_3a_1=4s_1^4-12s_2s_1^2+8s_3s_1,$$

where $s_k=\sum x_i^k$, $a_k=\sum x_1x_2\ldots x_k$. Thus is completed at last a very thorough account of symmetric functions of weights $\leqq 12$. Algebraists and statisticians for many years to come will have occasion to be thankful for the patience and perseverence of the authors. *D. H. Lehmer.*

Glowatzki, Ernst. Sechsstellige Tafel der Cauer-Parameter. Abh. Bayer. Akad. Wiss. Math.-Nat. Kl. (N.F.) no. 67, 37 pp. (1955).

These tables are of use in the design of electrical networks [cf. W. Cauer, Theorie der linearen Wechselstromschaltungen, Akademie Verlagsgesellschaft, Leipzig, 1941]. They give, for $\theta=0(1°)90°$ and for $m=1(1)n$, $n=1(1)12$, the values to 6D of $a_m=(\sin\theta)^{\frac{1}{2}}\,\mathrm{sn}\,(mK/n;\sin\theta)$, to 6S of Δ and to 3D of $-\ln\Delta$. Here, if n is odd, $\Delta=a_n^{-1}\prod_\nu a^2_{2\nu-1}$, where ν runs from 1 to $\frac{1}{2}(n+1)$, while if n is even $\Delta=\prod_\nu a^2_{2\nu-1}$, where ν runs from 1 to $\frac{1}{2}n$. When $n=1$, 2, 3, 5, 6, 9, 10 the computations were carried out by hand to 12D, using the tables of G. W. and R. M. Spenceley [Smithsonian elliptic functions tables, Washington, D. C., 1947; MR **9**, 380]. When $n=4$, 7, 8, 11, 12 the computations were carried out on the Göttingen automatic computer G1, using the series expansions, and working to 9D. No details of the checking are given. The tables are clearly printed.

The parameters a_m are the zeros of the Zolotareff functions of degree n, based on the interval I: $(-k^{1/2}, k^{1/2})$, where $0<k=\sin\theta<1$. The Zolotareff function is that rational function of x, with numerator and denominator of degree n, with value unity at $x=1$, which gives best approximation (in the sense of minimal deviation) to zero in the interval I and to ∞ in the complement of the interval $(-k^{-1/2}, k^{-1/2})$. The maximal deviation from zero in I is Δ. For an account of these functions see H. Piloty [Z. Angew. Math. Mech. **34**, 175–189 (1954); MR **15**, 871]. *John Todd.*

Horner, F. A table of a function used in radio-propagation theory. Proc. Inst. Elec. Engrs. C. **102**, 134–137 (1955).

The real and imaginary parts of $1+2i\omega^{1/2}e^{-\omega}\int_{-1\sqrt{\omega}}^{\infty}e^{-x^2}dx$ are tabulated to 3D for $\mathrm{Re}\,\omega$, $-\mathrm{Im}\,\omega=0(.5)10$, $\mathrm{Re}\,\omega$, $-\mathrm{Im}\,\omega=0(.1)2$ and $\mathrm{Re}\,\omega$, $-\mathrm{Im}\,\omega=0(.05)0.5$. The calculations were carried out on the Pilot Ace of the National Physical Laboratory. No description of method used, or checks made are given; references to related tables are given. The table has been spot checked by comparison with the tables of V. N. Faddeeva and N. M. Terent'ev [Tables of values of the function

$$w(z)=e^{-z^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}dt\right)$$

for a complex argument, Gostehizdat, Moscow, 1954; MR **16**, 960]. The only discrepancy noted was in rounding at $\omega=-8i$. *John Todd* (Washington, D. C.).

Mapother, Dillon E., and Snyder, James N. The axial variation of the magnetic field in solenoids of finite thickness. Engineering Experiment Station Circular no. 66, Univ. of Illinois, Urbana, Ill., 1955. 93 pp.

The tables in this report contain correction factors used in the design of solenoids; they take account of the effect, on the magnetic field of the solenoid, of the finite thickness of the winding. The tables have been computed on the Illiac

(University of Illinois high speed digital computer), and all tabular values are given to 6D. *A. Erdélyi.*

Hijikata, Katsunori. Tables of auxiliary functions useful for calculation of molecular integrals. Rep. Univ. Electro-Commun. 5, 35–42 (1953).

This paper contains numerical tables of $W_\tau^\nu(m, n; \alpha, \alpha)$ defined in a paper by Ishiguro [Nat. Sci. Rep. Ochanomizu Univ. 4, 64–76 (1953); MR 15, 255] and of $G_\tau^\nu(m; \alpha)$ defined in another paper [Ishiguro, Yuasa, Sakamoto, and Kimura, ibid. 4, 176–191 (1954), Table XXI; MR 16, 175]. In both tables $\alpha = 7.5$, 8, 8.5, and m, n, ν, τ run through an assortment of integers. *A. Erdélyi* (Pasadena, Calif.).

Todd, John. Motivation for working in numerical analysis. Comm. Pure Appl. Math. 8, 97–116 (1955).

The author claims [justifiably?] that "the profession of numerical analysis is not yet so desirable that it is taken up by choice . . . . Most of those . . . in this field have been, more or less, drafted into it . . . ." With a most enjoyable exposition of significant problem areas the author demonstrates "that numerical analysis is an attractive subject where mathematics of practically all sorts can be used significantly." This should be required reading for all numerical analysts and students of mathematics.

The problem areas include: number of multiplications needed to evaluate a polynomial; convergence acceleration; modified differences; bounds for eigenvalues; errors of quadrature formulas; game theory, assignment problems, etc.; Monte Carlo (very lightly touched on); combinatorial problems; adventures with electronic computers.

G. E. Forsythe (New York, N. Y.).

Todd, John. Begründung für die Beschäftigung mit numerischer Analysis. Jber. Deutsch. Math. Verein. 58, Abt. 1, 11–38 (1955).

Translation by E. Kamke of the paper reviewed above.

✶Salvadori, Mario G. Numerical methods in engineering. With a collection of problems by Melvin L. Baron. Prentice-Hall, Inc., New York, 1952. xiii+258 pp. $7.15.

This book is intended to introduce the elementary techniques of numerical methods which are needed in the solution of technical problems, and even in this restricted field, it only partially succeeds in its aim. The topics described are the following. (1) The solution of algebraic equations, principally by linear interpolation, and Newton's (and other's) methods of successive approximation. (2) The solution of simultaneous algebraic equations by the Gauss and Cholesky methods, by the iteration process of Gauss-Seidel, and by the method of relaxation. (3) Interpolation, the calculation of derivatives and the evaluation of definite integrals. Much of this chapter is based upon Taylor's series, and one looks in vain for a systematic exposition of finite differences. The interpolation formula considered is the Gregory-Newton; derivatives are expressed in Lagrange type formulae (using ordinates), and there is no examination in any of the examples of the neglected differences. Integrals are evaluated by the trapezoidal and Simpson rules; and to improve the simple approximations, Richardson's extrapolation process is used.

(4) The solution of ordinary differential equations with initial boundary conditions. After describing the use of Taylor's series to calculate the first few values, the Adam's-Bashforth formula using backward differences and the procedures described by Fox and Goodwin [Proc. Cambridge Philos. Soc. 45, 373–388 (1949); MR 10, 744] are explained.

(5) The solution of ordinary differential equations with two-point boundary conditions. The use of step-by-step methods with assumed end conditions is outlined, but no distinction is made between linear and non-linear equations. The other procedures described involve the replacement of the differential operators by their finite-difference equivalents, and the solution of the resulting set of simultaneous equations by standard methods or by relaxation including the difference corrections of Fox. Richardson's extrapolation is used to improve the accuracy. The solution of characteristic-value problems is indicated.

(6) The solution of partial differential equations (predominately of the elliptic kind). The finite difference representations of the differential operators are established, and these are used in the differential equations (Poisson's and the biharmonic) to obtain a set of simultaneous equations which are solved by direct methods or by relaxation, as (5) above. Curved boundaries with known boundary values, are treated, and the plastic-torsion problem and characteristic-value problems are considered. One example shows the solution of a heat-flow problem in two dimensions, and the use of other coordinates and triangular nets is described.

D. C. Gilles (Manchester).

✶Orloff, Constantin. The fundamentals of practical spectral arithmetic and algebra. Instructions for a new method of calculation on calculating machines. Libertatea, Yugoslavia, 1955. 46 pp.

Spectral arithmetic has been discussed by the author [e.g. Bull. Soc. Math. Phys. Serbie 5, no. 1–2, 17–30 (1953); MR 15, 472]. Detailed instructions for using it and extensions on desk calculators are given here. The basic ideas are well-known to experienced computers: see, for instance, L. J. Comrie's procedure for computing $\sum a_i p_i$, where a_i are integers and p_i are amounts set in sterling (pounds, shillings and pence) [cf. D. R. Hartree, Numerical analysis, Oxford, 1952, p. 275; MR 14, 690]. *John Todd.*

Blanc, Ch., und Liniger, W. Stochastische Fehlerauswertung bei numerischen Methoden. Z. Angew. Math. Mech. 35, 121–130 (1955). (English, French and Russian summaries)

Dans des publications antérieures [Comment. Math. Helv. 26, 225–241 (1952); Arch. Math. 5, 301–308 (1954); MR 14, 691; 16, 290] le premier des deux auteurs a montré comment on peut comparer entre elles diverses méthodes de calcul approché en considérant les données comme aléatoires et en étudiant les propriétés statistiques de l'erreur. Le présent mémoire donne des résultats détaillés pour les formules de quadrature approchée et les problèmes aux limites différentiels. Cette étude fait apparaître pour le premier problème les avantages de la formule de Simpson, pour le deuxième la supériorité des méthodes aux différences sur les méthodes d'approximation fonctionnelle.

J. Kuntzmann (Grenoble).

Fujita, Hiroshi. Contribution to the theory of upper and lower bounds in boundary value problems. J. Phys. Soc. Japan 10, 1–8 (1955).

This paper presents a method of estimating the bounds of errors in approximate methods in boundary-value problems. The method is based on the theory of the self-adjoint operator $H = T^*T$ of Hilbert space as developed by von Neu-

mann and others. Some examples are given showing how some of the linear operators of mathematical physics may be expressed in this form. Theorems are presented which give a method for obtaining upper and lower bounds at a point for a boundary-value problem. The methods used are essentially not different from those which occur throughout the literature but are interesting because it presents the connection between this type of problem and theory of Hilbert spaces.

A numerical example considers Poisson's equation in a square with the condition that the solution vanish on the boundary. The second-order approximation produces very satisfactory results in this instance. *C. G. Maple.*

★**Vernotte, Pierre. L'interpolation idéale et son calcul numérique effectif.** Publ. Sci. Tech. Ministère de l'Air, no. 300. Paris, 1955. v+74 pp. 850 francs.

Salzer, Herbert E. Osculatory interpolation in the complex plane. J. Res. Nat. Bur. Standards **54**, 263–266 (1955).

L'auteur donne dans certains cas précis les coefficients de l'interpolation d'Hermite d'une fonction analytique. Les valeurs de la variable pour lesquelles on suppose connues la fonction et sa dérivée sont les suivantes: (0, 0 et 1, 0), (0, 0; 0, 1; 1, 0), (0, 0; 0, 1; 1, 0; 1; 1), (0, 0; 0, 1; 1, 0; 1, 1; 2, 0), (0, 0; 0, 1; 0, 2; 1, 0; 1, 1; 2, 0), (0, 0; 0, 1; 0, 2; 1, 0; 1, 1; 2, 0; 2, 1). *J. Kuntzmann* (Grenoble).

Stegun, Irene A., and Abramowitz, Milton. Generation of Coulomb wave functions by means of recurrence relations. Phys. Rev. (2) **98**, 1851–1852 (1955).

The regular and irregular Coulomb wave functions, F_L and G_L, have the property that $F_L\to 0$, $G_L\to\infty$ as $L\to\infty$. Thus, in using recurrence relations for the computation of these functions, in the case of the irregular Coulomb wave functions one may start with G_0 and G_1, and generate the G_L with increasing L; in the case of the regular Coulomb wave functions it would seem necessary to start with functions of the highest order, and use the recurrence relations with decreasing L. The authors now describe a method which avoids direct computation of F for the highest values of L. Briefly, their method consists in starting with a linear combination of F_L and G_L which has arbitrarily prescribed values for the highest L, say for $L=N$ and $N+1$. $y_L=\alpha F_L+\beta G_L$, $y_N=1$, $y_{N+1}=0$ leads to $y_L=\alpha(F_L-F_{N+1}G_L/G_{N+1})$, where $\alpha=G_{N+1}/(F_NG_{N+1}-F_{N+1}G_N)=G_{N+1}[(N+1)^2+\eta^2]^{1/2}/(N+1)$. Now, for L small in comparison with N, $y_L=\alpha F_L$ approximately, and this can be used for the computation of the F_L. As an example, the authors choose $N=30$, and compute F_L for $L=0$, 1, 10, 11, 20 to 8S. *A. Erdélyi.*

Craig, Edward J. The N-step iteration procedures. J. Math. Phys. **34**, 64–73 (1955).

The paper (taken from the author's January 1954 M.I.T. dissertation) reviews the basis of the N-step iterative processes of Hestenes, Lanczos, and Stiefel [see Hestenes and Stiefel, J. Res. Nat. Bur. Standards **49**, 409–436 (1953); MR **15**, 651], and offers another one of his own. [Craig's process was suggested independently by Hestenes at the American Mathematical Society Sixth Symposium on Applied Mathematics, Santa Monica, August 1953; see the Proceedings, in press.]

To solve $Ax=y$ in N-space, the basic idea is to find N independent vectors $p_0, \ldots, p_{N-1}$ which are B-orthogonal (i.e., $p_j{}^*Bp_k=0$, $j\neq k$) with respect to some symmetric (hermitian in the complex case) matrix B. If $e_0=x_0-A^{-1}y$, then $A^{-1}y=x_0-\sum_{j=0}^{N-1} m_jp_j$, where $m_j=p_j{}^*Be_0/p_j{}^*Bp_j$. Theoretically B need not be definite if no $p_j{}^*Bp_j=0$. The problem in designing gradient methods is to choose B so related to A that the $p_j{}^*Be_0$ can be computed without knowing e_0 or $A^{-1}y$. The above authors took $B=A$, if $A=A^*$, and $B=A^*A$ otherwise. Craig's suggestion for a general nonsingular A is to choose a basis $\{b_j\}$ first, let $p_j=Ab_j$, and take $B=I$. His algorithm finally takes the form: $\epsilon_{-1}=0$, x_0 arbitrary, $r_k=Ax_k-y$, $m_k=|r_k|^2/|A^*b_k|^2$, $x_{k+1}=x_k-m_kA^*b_k$, $\epsilon_{k-1}=|r_k|^2/|r_{k-1}|^2$, $b_k=r_k+\epsilon_{k-1}b_{k-1}$. The author asserts that with automatic computing machines the N-step methods work well if the machine (base β) carries more digits than $\frac{1}{2}\log_\beta(\max\lambda/\min\lambda_i)$, where the λ_i are eigenvalues of A^*A. *G. E. Forsythe* (New York, N. Y.).

Ostrowski, A. M. On the linear iteration procedures for symmetric matrices. Rend. Mat. e Appl. (5) **14**, 140–163 (1954).

This is a discussion of convergence questions for generalizations of the "cyclic single step" iterative method for solving a linear system $A\xi'=\eta'$, where A is symmetric. (The customary name 'Seidel process' is rejected by the author because, surprisingly, Seidel advised not using the method in his oft-cited paper [Abh. Bayer. Akad. Wiss. Math.-Phys. Kl. 11, 81–108 (1874)].) The generalizations are in the following directions: (1) Let g denote the subscripts in a group. Let $\xi=(x_\alpha)$, $\eta=(y_\alpha)$. New values x_α ($\alpha\in g$) are determined from the old values x_α (α non-$\in g$) by solving the linear system

$$(*)\qquad \sum_{\beta\in g} a_{\alpha\beta}x_\beta=y_\alpha-\sum_{\gamma\ \text{non-}\in G} a_{\alpha\gamma}x_\gamma\quad(\alpha\in g).$$

(2) The relaxation from the old values to the new ones may be "incomplete", i.e., one may under- $(0<q_k<1)$ or over-relax $(1<q_k<2)$ the kth group. (3) A is permitted to be slightly unsymmetric. (4) In the various groups the order of relaxing ("steering") is permitted to vary. For example, in the single-step case, one may follow roughly the rules of Gauss [Werke, vol. 9, Teubner, Leipzig, 1903, pp. 278–281; translated by the reviewer, Math. Tables Aids Comput. **5**, 255–258 (1951)], Seidel [loc. cit.], or Southwell [Proc. Roy. Soc. London. Ser. A. **151**, 56–95 (1935)].

The basic theorems are: I. If A is Hermitian positive definite, the "cyclic linear group iteration" converges for any η, for any start ξ_0, and for any factors q_k, so long as $0<\epsilon\leqq g_k\leqq 2-\epsilon$, for all k. II. Suppose A is regular and Hermitian indefinite, but that the diagonal block $(a_{\alpha\beta})$ of $(*)$ is positive definite for each group. Then for each η there are open sets of ξ_0 such that the cyclic linear group iteration diverges for all choices of q_k in the interval $(0, 2)$. (The proof of II takes four lines.) For the cyclic single-step case and $q_k\equiv 1$, result II is due to Reich [Ann. Math. Statist. **20**, 448–451 (1949); MR **11**, 136].

The remaining results of Ostrowski's paper show the convergence to be linear and estimate the constant, in various cases of (1), (2), (3), (4). By means of an example the "tactics" of steering for short-run gain are compared with a "strategy" for long-run gain in speed of convergence. *G. E. Forsythe* (New York, N. Y.).

Stiefel, Eduard. Ausgleichung ohne Aufstellung der Gaussschen Normalgleichungen. Wiss. Z. Tech. Hochsch. Dresden **2**, 441–442 (1953).

In applying the author's method of conjugate gradients [M. R. Hestenes and E. Stiefel, J. Res. Nat. Bur. Standards **49**, 409–436 (1953); MR **15**, 651] to solve a linear system

of form (*) $A^TAx+A^Tl=0$, it is not necessary actually to form (*). One can instead multiply certain vectors by A and by A^T as they arise. But the normal equations for minimizing $|Ax+l|^2$ (A has m columns and $n>m$ rows) take the form (*). The resulting algorithm for adjusting the data in a linear least-squares problem is explained in geodetic notation for n inconsistent equations in 3 unknowns, with a numerical example for $n=5$. *G. E. Forsythe.*

Vzorova, A. I. On the solution of a system of linear algebraic equations by Yu. A. Šreĭder's method. Vyčisl. Mat. Vyčisl. Tehn. 1, 90–94 (1953). (Russian)

In Šreĭder's method [Dokl. Akad. Nauk SSSR (N.S.) **76**, 651–654 (1951); MR **12**, 639], one starts with a Gram-Schmidt process to orthogonalize the rows of the matrix of coefficients. The author tested the method on two numerical matrices of order 10. She concludes that the method is very convenient for well conditioned problems, but that the loss of precision (approximately the same as that with Gaussian elimination) in badly conditioned problems complicates the work. A compact data sheet arrangement is included. *G. E. Forsythe* (New York, N. Y.).

Couffignal, Louis. Méthodes pratiques de réalisation des calculs matriciels. Rend. Mat. e Appl. (5) 14, 85–97 (1954).

The paper discusses the precision of solving a system of linear algebraic equations (*) $Ax=b$. The author distinguishes "physical" uncertainties, due to uncertainties in A, b, from the "calculational" errors of an algorithm computing $A^{-1}b$, and gives expressions for both. The author looks at some ill-conditioned matrices and concludes that, whenever (*) is the mathematical translation of a physical problem, one should avoid any solution process involving progressive reduction of the residual $r=b-Ax$, including relaxation methods and use of analog machines. The author favors a form of Gaussian elimination. *G. E. Forsythe.*

Curtiss, J. H. A theoretical comparison of the efficiencies of two classical methods and a Monte Carlo method for computing one component of the solution of a set of linear algebraic equations. Institute of Mathematical Sciences, New York University, Rep. **IMM-NYU 211**, ii+56 pp. (1954).

The equations of the title are $\xi=H\xi+\gamma$, where

$$h=\|H\|=\max_i \sum_{j=1}^{n} |h_{ij}|<1.$$

The two classical methods of the title are (a) Gaussian elimination, and (b) the stationary linear iteration $\xi_N=H\xi_{N-1}+\gamma$ ($N=0, 1, \cdots$). The Monte Carlo method (c) is a statistical sampling of a certain random variable whose expectation is (*): $\xi_N=\xi_0+(I+H+\ldots+H^{N-1})\rho_0$, where $\rho_0=\gamma-(I-H)\xi_0$, for some fixed large N. Using partitioned matrices, the author rephrases $\xi_N-\xi_0$ in (*) as part of a certain vector $K^N\theta$. To estimate $K^N\theta$ he forms a random walk of N steps with transition probabilities mostly equal to $|h_{ij}|$, and with weights mostly ±1. [See Curtiss, J. Math. Phys. **32**, 209–232 (1954); MR **15**, 560.]

The author gives formulas and tables for the number of multiplications required by the three methods to make $\|\xi_\infty-\xi_N\|<r\|\xi_\infty-\xi_0\|$, (taking no advantage of zeros in H) as functions of n, h, and r. The reviewer's rough summaries of these are: (a) $n^3/3$; (b) $(\log r)(\log h)^{-1}n^2$; (c) $n^2+2(1-h)^{-2}r^{-2}$. Looking at the formulas for variable n but fixed h [is this realistic?], the author concludes that (c) is most efficient for large enough n. A sample of the author's table shows that for $h=.9$, $r=.01$, (c) prevails for $n\geqq555$. There is some discussion of matrix inversion also.
G. E. Forsythe (New York, N. Y.).

Egerváry, E. Über die Faktorisation von Matrizen und ihre Anwendung auf die Lösung von linearen Gleichungssystemen. Z. Angew. Math. Mech. 35, 111–118 (1955). (English, French and Russian summaries)

The abbreviated Gaussian elimination processes for linear systems are theoretically based on the Banachiewicz triangular decomposition of a square matrix $\mathfrak{A}$ into $\mathfrak{L}\mathfrak{U}$, where $\mathfrak{L}$, $\mathfrak{U}$ are lower, upper triangular, respectively. [Reviewer's note: See, e.g., A. M. Turing, Quart. J. Mech. Appl. Math. **1**, 287–308 (1948); MR **10**, 405.] That decomposition requires that no upper-left principal minor determinant vanish, so that $\mathfrak{A}$ may not be singular. The author extends the basic ideas to systems $\mathfrak{A}\mathfrak{x}=0$, where $\mathfrak{A}$ is an arbitrary rectangular matrix. By matrix methods involving no determinants he finds a decomposition $\mathfrak{A}=\mathfrak{B}\mathfrak{C}$, where $\mathfrak{B}\mathfrak{C}\mathfrak{x}=0$ implies $\mathfrak{C}\mathfrak{x}=0$, where $\mathfrak{C}$ has the same rank and number of rows as $\mathfrak{A}$, and where $\mathfrak{C}$ is an upper triangular form suitable for the recursive solution of $\mathfrak{C}\mathfrak{x}=0$. It is noted that a nonhomogeneous system in n variables can always be reduced to a homogeneous system in $n+1$ variables, so that the above result suffices for all systems $\mathfrak{A}\mathfrak{x}=\mathfrak{b}$; a consistency condition is given. There is a leisurely discussion followed by three numerical examples. *G. E. Forsythe.*

Abramov, A. A. On the influence of round-off errors in the solution of Laplace's equation. Vyčisl. Mat. Vyčisl. Tehn. 1, 37–40 (1953). (Russian)

The author considers the simple iterative algorithm $x^{(k+1)}=Ax^{(k)}+b$ for solving a linear system $(E-A)x=b$, where A is a symmetric matrix with $\|A\|<1$. Because of round-off errors one really computes $y^{(k+1)}$, where $z^{(k+1)}=y^{(k+1)}-x^{(k+1)}=\epsilon^{(k)}+A\epsilon^{(k-1)}+\cdots+A^k\epsilon^{(0)}$. Assume $\epsilon^{(0)}$, $\cdots$, $\epsilon^{(k)}$ are independent random vectors with zero mean and identical dispersion matrices σ^2E, where E=identity. Then $Dz^{(k+1)}\approx\sigma^2(E-A^2)^{-1}$.

In the solution of Laplace's difference equation in k dimensions by simple iteration (equivalent to solving the heat equation by difference methods), A is the averaging operator for the $2k$ neighbors of a point. For $k=2$ over a rectangular net region of Np_1 by Nq_1 points, the author gives bounds for the largest element C of $(E-A^2)^{-1}$, and notes that as $N\to\infty$, $c\approx(2/\pi)\log N$. For $k>2$, the corresponding quantity is bounded as $N\to\infty$. The author concludes that round-off has little influence on this algorithm.
G. E. Forsythe (New York, N. Y.).

Lyusternik, L. A. On convergence of an iterative process of solution of a system of algebraic equations for random initial data and accumulation of errors. Vyčisl. Mat. Vyčisl. Tehn. 1, 41–45 (1953). (Russian)

A system $y=Ay+b$, where A is symmetric and of order n, is solved by the iterative process $y_k=Ay_{k-1}+b$. Assume the components $\epsilon_0^{(j)}$ of the initial error $\epsilon_0=y_0-(I-A)^{-1}b$ to be independently distributed with mean 0 and dispersion D. Let ϵ_k be the error in y_k. If the eigenvalues of A are λ_i, it is shown that the mathematical expectation μ of $n^{-1}\sum_{j=1}^n(\epsilon_k^{(j)})^2$ is $Dn^{-1}\sum_{i=1}^n\lambda_i^{2k}$. When $I-A$ is the Laplacian finite-difference operator over a rectangle in $\nu=1$ or 2 dimensions, the author gives the approximate expressions $\mu\approx(\pi k)^{-\nu/2}$, valid when $k\ll n$, but too high when $k\gg n$.

He next loosely discusses the effect of round-off error for $\nu=2$, supposing new independent errors like ϵ_0 to be introduced at each step of the iteration. This suggests a round-off error growing like $\ln k$. However, a comparison with the analysis of Abramov [see the preceding review] shows the author that $\ln k$ is indeed too high, and should be replaced for $k>n$ by $\ln n$. The case $\nu=3$ is mentioned also.

G. E. Forsythe (New York, N. Y.).

Rutishauser, Heinz, et Bauer, Friedrich L. Détermination des vecteurs propres d'une matrice par une méthode itérative avec convergence quadratique. C. R. Acad. Sci. Paris **240**, 1680–1681 (1955).

The authors announce an accelerated convergence of Rutishauser's algorithm for computing the eigenvalues (assumed distinct) of a real symmetric matrix [same C. R. **240**, 34–36 (1955); MR **16**, 785]. The new algorithm follows. For $k=1, 2, \ldots$, let Λ_k be lower triangular with 1's on the diagonal, let D_k be diagonal, and let $\Sigma_k=\Lambda_k^T$. Given Λ_k, D_k, Σ_k $(\Lambda_1D_1\Sigma_1=A)$, decompose $\Sigma_k\Lambda_k=\Lambda_k^{**}D_k^{**}\Sigma_k^{**}$, and obtain $\Lambda_{k+1}=\Lambda_k(D_k\Lambda_k^{**}D_k^{-1})$, $\Sigma_{k+1}=\Lambda_{k+1}^T$, $D_{k+1}=D_k^2D_k^{**}$. It is stated that $\Lambda_k\to\Lambda_\infty$ quadratically, and that

$$\Lambda_\infty x_s\ (s=1, \ldots, n)$$

are the eigenvectors of A, where x_s are the eigenvectors of the triangular matrix Λ_∞. The connection with the earlier paper is that $A_1\Lambda_k=\Lambda_kA_{2^k+1}$. *G. E. Forsythe.*

Kreyszig, E. Die Einschliessung von Eigenwerten hermitescher Matrizen beim Iterationsverfahren. Z. Angew. Math. Mech. **34**, 459–469 (1954). (English, French and Russian summaries)

The basic problem is to obtain precise bounds for the eigenvalues of a hermitian matrix out of a matrix iteration procedure. Following Wielandt [Math. Z., in press] the author defines an "inclusion interval" for the "p-step method" as a closed interval $G=G(x_0, x_1, \ldots, x_p)$ of the projective line which intersects the spectrum of any hermitian matrix for which $x_1=Ax_0, \ldots, x_p=Ax_{p-1}$. A "minimal inclusion interval" has no subinterval which is an inclusion interval. While Wielandt [loc. cit.] has described such intervals for arbitrary p, the author claims that only the methods $p=1$ and 2 are practicable numerically, because of the labor involved for $p>2$.

The author summarizes Wielandt's theory for $p=1$, 2, and gives exact and approximate, geometrical and analytical methods for constructing the intervals in terms of the "moments" (Schwarz constants) $m_\nu=(x_k, x_{\nu-k})$. Shortest minimal inclusion intervals are treated. Since subtractions of like numbers occur in the computations, the author advises modifying the iteration to obtain three orthogonal vectors $z_0=x_0$, $z_1=Az_0-a_{10}z_0$, $z_2=Az_1-a_{21}z_1-a_{20}z_0$. There are two pictures and three numerical examples: with a matrix operator, an ordinary differential operator, and an integral operator.

In formula (1) Λ_2 should be corrected to read Λ^2.

G. E. Forsythe (New York, N. Y.).

Kreyszig, E. Die Ausnutzung zusätzlicher Vorkenntnisse für die Einschliessung von Eigenwerten beim Iterationsverfahren. Z. Angew. Math. Mech. **35**, 89–95 (1955). (English, French and Russian summaries)

This is a sequel to the paper reviewed above on bounding the eigenvalues of hermitian matrices when s iterates $y_0, Ay_0, \ldots, A^sy_0$ are known. It is now assumed that we have certain external information about the eigenvalues, e.g., that all eigenvalues are positive, or that $\lambda_3\geq a$, etc. The author shows how to use this information in locating minimal inclusion intervals for eigenvalues. The theorems are applied for $s=1$, 2 the three examples of the paper cited, using semidefiniteness of the operators as the external information. *G. E. Forsythe* (New York, N. Y.).

Granat, Yu. L. An iterative scheme of computing the roots of algebraic equations of high degree and the construction of transfer processes. Inžen. Sb. **20**, 168–176 (1954). (Russian)

This is a method for finding roots of algebraic equations by iteration, based essentially on factoring the given polynomial into two polynomial factors. In practice one of these would normally be of first degree (for a real root) or second degree (for a pair of complex roots). The interesting feature is the process for carrying out the iterations leading to the factorization. Aside from computational short-cuts and occasional difficulties the essential steps are as follows.

Let $F(x)$ be the given polynomial translated (if needful) to put all roots in one half-plane. Let $f_0(x)$ be a trial factor with degree lower than that of $F(x)$, and with leading coefficient unity. Divide $F(x)$ by $f_0(x)$, both being arranged in descending powers. Let $\varphi_0(x)$ denote the integral part of this result, arranged in ascending powers and divided by the constant term so that the first coefficient is unity. Arrange $F(x)$ in ascending powers, and divide by $\varphi_0(x)$. Let $f_1(x)$ denote this result, truncated to the same degree as $f_0(x)$ and divided by the coefficient of the term of highest degree. This completes the first cycle of the iteration. The second cycle repeats the same steps with $f_1(x)$ in the role of $f_0(x)$, and so ad infinitum. The author devotes considerable attention to a compact scheme of calculation. *W. E. Milne.*

Berger, E. R. Die Wurzeln der charakteristischen Gleichung für das kreiszylindrische Rohr. Ing.-Arch. **22**, 156–159 (1954).

The displacement is assumed to be of the form

$$\exp(-\lambda x/a)\cdot\exp(im\phi).$$

One then has a characteristic equation which is biquadratic in λ^2 as well as in m^2. Physical considerations require m to be integral, so that interest centers on the solution for λ^2, up to now only crudely accomplished, according to the author. The method calls for the substitution

$$\lambda^2=y\gamma^{-1}[n(n-\gamma)]^{1/2},$$

where $n=\gamma m^2$, and γ is an elastic constant. A quadratic equation in $y+y^{-1}$ results, except for a negligible term in $y-y^{-1}$. *R. E. Gaskell* (Seattle, Wash.).

Duijvestijn, A. J. W., and Berghuis, J. The computation and the expansion of some triple integrals originating from the theory of cosmic rays. Math. Centrum Amsterdam, Rekenafdeling, rep. **R 261**, 17 pp. (1955).

The evaluation of a triple integral

$$K=\int\int\int rB^{-5/2}(1-e^{-B\alpha})(1-e^{-B\beta})(1-e^{-B\gamma})drdBd\theta,$$

where α, β, γ are certain explicit functions of r, θ, involving a parameter, is discussed in detail. The ranges are $0\leq\theta\leq\frac{1}{2}\pi$, $0\leq B<\infty$, $0\leq r<\infty$. Asymptotic expansions of the indefinite integrals occurring are obtained and these integrals are tabulated. Six values of the parameter are discussed for each of two representations of α, β, γ. *John Todd.*

Karmazina, L. N. On a method of computation of the hypergeometric function. Vyčisl. Mat. Vyčisl. Tehn. **2**, 111–115 (1955). (Russian)

In order to compute $F(a, b, c, z)$, the author proposes the numerical evaluation of the integral

$$\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-xz)^{-a}dx$$

by Gauss-Jacobi mechanical quadrature based on the orthogonal polynomials associated with the weight function $x^{b-1}(1-x)^{c-b-1}$ [cf. Karmazina, MR **16**, 959]. The paper contains estimates of the error term, and 6D tables of the Christoffel numbers and the zeros of the orthogonal polynomials for $n=5$, $b=.1(.2).9$, $c=2.1(.1)3$.

A. Erdélyi.

Lombardi, Jose P. Application of the Picard-Lindelöf iteration method to the solution of differential equations. Univ. Nac. Eva Peron. Publ. Fac. Ci. Fisicomat. no. **206**, Serie Tercera. Publ. Esp. **43**, 117–136 (1953). (Spanish. English summary)

The author considers a system of four first-order equations associated with the transverse movements of an airplane in flight. Three of these are linear but the fourth contains sin φ, where φ is a dependent variable. To solve this system the author replaces sin φ by φ and solves the resulting linear system by Heaviside's method. With this solution as a start he then uses the Picard-Lindelöf method of iteration to secure improved solutions of the non-linear system. Coefficients needed in the process are calculated to six figures for the first four approximations and the solution is given to five figures through the second approximation.

W. E. Milne (Corvallis, Ore.).

Ionescu, D. V. Une généralisation d'une propriété intervenant dans la méthode de Runge-Kutta pour l'intégration numérique des équations différentielles. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 229–241 (1954). (Romanian. Russian and French summaries)

As is well known, the Runge-Kutta method for integrating the equation $y'=\varphi(x, y)$ over an interval of length h is based on the construction of an auxilliary function $K(h)$ whose development in powers of h agrees through the first four terms with that of the actual solution. The author gives a generalization of this procedure by which he can obtain a $K(h)$ for which the development agrees through the first m terms.

W. E. Milne (Corvallis, Ore.).

Vogel, W. Ein graphisches Verfahren zur Lösung von linearen Differentialgleichungen zweiter Ordnung. Ing.-Arch. **23**, 119–121 (1955).

The paper describes a graphical construction for the solution of the second-order differential equation

$$f_1(x)y''+f_2(x)y'+y=g(x)$$

with end-point boundary conditions. The construction determines y at equidistant points 1, 2, 3, using the equation written in the form

$$g(x)=y+\frac{\Delta y_2}{\Delta x}(a+f_2(x))-\frac{\Delta y_1}{\Delta x}a,\quad a=\frac{f_1(x)}{\Delta x}-\frac{f_2(x)}{2},$$

the derivatives being expressed in their simple difference form.

D. C. Gilles (Manchester).

Rapoport, I. M. On an approximate method in a one-dimensional boundary problem. Ukrain. Mat. Ž. **6**, 202–217 (1954). (Russian)

The principal feature of the method is the use of a special type of approximating polynomials to represent the solution over successive sub-intervals. Formulas are derived for the determination of these polynomials in terms of the sought-for solution and its successive derivatives. The approximate characteristic values and characteristic solutions are found from the latent roots and latent vectors respectively of a finite matrix set up with the aid of the above mentioned polynomials.

W. E. Milne (Corvallis, Ore.).

Šamanskiĭ, V. E. An approximate method of solution of the Dirichlet problem for the Laplace equation. Dokl. Akad. Nauk SSSR (N.S.) **100**, 1049–1052 (1955). (Russian)

The author presents a method for construction of an approximate solution of the Dirichlet problem for the equation $\Delta\varphi=\varphi_{xx}+\varphi_{yy}=0$, in a domain D which can be separated by a simple arc γ into two domains D_1 and D_2, in each of which the construction of a solution does not present special difficulty. The method depends on the use of a complete orthonormal system $\{\varphi_k(s)\}$ defined on γ. The coefficients C_k of an Nth partial sum $\sum_1^N C_k\varphi_k(s)=P_N(s)$ are to be chosen so as to minimize the functional

$$\int_\gamma (\partial\Phi_N^-/\partial n-\partial\Phi_N^+/\partial n)ds$$

where $\Phi_N(x, y)$ is the function harmonic in D_1 and D_2, equal to $P_N(s)$ on γ and assuming the prescribed values on the boundary Γ of D. The Nth approximation is thus obtained, as in the Rayleigh-Ritz procedure, as the solution of an Nth order system of linear equations.

Conditions for convergence are given. It is sufficient, in particular, that all boundary data be sufficiently smooth and that the angles made by γ and Γ at their points of intersection lie between $\pi/4$ and $3\pi/4$.

R. Finn.

Filin, A. P. Solution of integral equations by means of the scale of centers of gravity. Inžen. Sb. **20**, 177–182 (1954). (Russian)

The solution of Fredholm's integral equation of the second kind by means of successive approximations requires the evaluation of a definite integral at each step of the approximation. The author proposes to accomplish these integrations by the aid of A. A. Popov's method of the scale of centers of gravity [A new method of integration by means of orthogonal foci, Gostehizdat, 1947; Quart. Appl. Math. **3**, 166–174 (1945); MR **7**, 86].

W. E. Milne.

Kahn, F. D. The correction of observational data for instrumental band width. Proc. Cambridge Philos. Soc. **51**, 519–525 (1955).

Let $\phi(x)$ denote the readings, given by a measuring instrument, of a true function $f(x)$. The two functions are related to the apparatus function $q(x)$ by the convolution integral equation $\phi(x)=\int_{-\infty}^{\infty} g(u)f(x-u)\,du$. A new approximate method is presented for solving this equation for $f(x)$, when $g(x)$ is an even function with finite moments through the sixth order. The method is based on this lemma: When $g(x)$ denotes any function that has zero moments through the order k and $f(x)$ denotes any function with finite moments up to that order, then the convolution $\phi(x)$ of those two functions has zero moments up to that order. The

author uses the Dirac delta "function" $\delta(x)$ freely in the development of his method. His approximation $F(x)$ to $f(x)$ is represented by the convolution of $\phi(x)$ and $q(x)$, where $q(x)=\alpha_0\delta(x)+\sum_{n=1}^{\infty}\alpha_n[\delta(x+nh)+\delta(x-nh)]$, where h^2 is the ratio of the fourth and second moments of $g(x)$, and the numbers α_n are algebraic functions of those two moments. Maximum errors $|F(x)-f(x)|$ are examined. The approximation formula is applied in a particular case. The effect of observational errors in measuring $\phi(x)$ is discussed.

R. V. Churchill (Ann Arbor, Mich.).

✶Malengreau, Julien. Considérations sur le système de numération binaire et ses applications aux machines mathématiques d'intérêt purement scientifique. IIIe Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 71–75. Fédération belge des Sociétés Scientifiques, Bruxelles.

The author describes a small electromechanical demonstration model machine which finds the exponent of 2(mod p) provided p is a given prime of the form $2^n\pm2^m\pm1$. More generally, x and y are found such that $2^x\pm N=py$, where p and N are given integers with p limited as provided above and N is an odd integer $<2^{16}$.

D. H. Lehmer.

Burks, Arthur W. The logic of programming electronic digital computers. Indust. Math. 1, 36–52 (1950).

The paper at hand is an expository one presented by the author to introduce to his audience some of the problems arising in connection with machine computation. Once a mathematical problem has been formulated in terms which are suitable for digital machines, i.e. in a finite form, the set of instructions which characterizes this formulation must be prepared. The author is concerned with this matter of programming of problems. He gives a quite complete description of the problem together with some simple illustrative material. *H. H. Goldstine* (Princeton, N. J.).

Fil'čakov, P. F., and Pančišin, V. I. The electro-integrator EGDA-3. Ukrain. Mat. Ž. 7, 112–120 (1955). (Russian)

Diprose, K. V. Analogue computing in aeronautics. J. Roy. Aero. Soc. 59, 479–488; discussion, 488–493 (1955).

✶Nevskiĭ, B. A. Spravočnaya kniga po nomografii. [Reference book on nomography.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 376 pp. 10.70 rubles.

This book is designed to guide those concerned with the practical aspects of nomography. The range of questions raised, the abundance of specific instructions, the detailed index and table of contents, the adequacy of cross-references, the grouping of articles into sections and chapters all contribute to accomplishing this purpose. References to other works are infrequent.

The organization of material and classification of nomographs is based on the concept of contact between elements of movable planes. This is developed in the second chapter which contains an extensive table of structural forms. The other early chapters are concerned with scales, binary fields, characteristics, matters relating to accuracy, functions of two variables, intersection nomographs for functions of three variables, and anamorphosis (case of Saint Robert). The seventh chapter gives the canonical forms for alignment nomograms. These and the forms for movable planes are treated in later chapters by ordinarily giving dimensioned schematics, equations for scales with necessary parameters, a sequence of steps to be followed for computation and construction, and a fully explained example. The scope of the book may be further inferred from the remaining principal topics: the networks and frameworks of M. V. Pentkovskiĭ, approximate methods (the Lafay method is not given), use of repeated variables in alignment charts, binary scales, composite nomographs, three-dimensional alignment charts, and various complementary devices for increasing the utility of a nomograph. *R. Church* (Monterey, Calif.).

✶Newski, B. A. Praktikum der Nomogrammkonstruktionen. Akademie-Verlag, Berlin, 1955. xv+316 pp. DM 35.00.

Translation of the book reviewed above.

ASTRONOMY

✶Chazy, Jean. Mécanique céleste. Equations canoniques et variation des constantes. Presses Universitaires de France, Paris, 1953. vii+270 pp. 1260 francs.

Questa monografia deve essere considerata come un importante complemento dei due grandi trattati di meccanica celeste di Tisserand [Traité de mécanique céleste, t. I–IV, Gauthier-Villars, Paris, 1889, 1891, 1894, 1896] e Poincaré [Les méthodes nouvelles de la mécanique céleste, t. I–III, ibid., 1892, 1893, 1899; Leçons de mécanique céleste t. I–III, ibid., 1905, 1907, 1910]. L'esposizione magistrale dei concetti fondamentali realizza un felice raccordo fra le concezioni classiche e quelle relativistiche. L'A. procede al calcolo dello spostamento del perielio dei pianeti e della deflessione della luce nel campo gravitazionale del Sole e confronta i risultati teorici con le osservazioni.

I teoremi generali della teoria delle equazioni canoniche e della teoria delle perturbazioni sono presentati in forma impeccabile, tenendo conto di tutti i perfezionamenti che le teorie stesse hanno subito dopo Jacobi e Poincaré. Lo stesso autore inquadra nei luoghi opportini i contributi personali ben noti ai cultori della materia, rendendo ovunque molto interessante l'esposizione.

La monografia contiene una discussione dettagliata delle traiettorie d'una massa infinitesima e della propagazione della luce nel campo gravitazionale di una massa sferica omogenea caraterizzato dal ds^2 di Schwarzschild.

G. Lampariello (Messina).

Orlov, A. A. Almost circular periodic motions of a material point under the action of the Newtonian attraction of a spheroid. Moskov. Gos. Univ. Soobšč. Astr. Inst. no. 88-89, 3–38 (1953). (Russian)

Consider the motion of a particle of negligible mass under the Newtonian attraction of a nonhomogeneous spheroid of rotation, symmetric with respect to the plane perpendicular to its axis, and assume that the spheroid is not subject to the attraction of the particle. S. D. Černyĭ [Byull. Inst. Teoret. Astr. 4, no. 6 (59), 287–308 (1949); MR 12, 365] treated a particular case of this problem when the particle moves in the plane of the equator of the spheroid and the force function U of the attraction field of the spheroid is of the form $U=\mu_1/r+\mu_2/r^3$, where r is the distance of the particle from the center of the spheroid and μ_1, μ_2 are certain

constants. In this particular case the problem is integrable in terms of elliptic functions.

For the general case G. N. Dubošin [Moskov. Gos. Univ. Trudy Gos. Astr. Inst. 15, 158–250 (1945)] proved the existence of two families of periodic solutions which are close to circular motions. One of these families corresponds to the motion of the particle in a plane and the other family to its motion in space. In the case of the motion of a particle under gravitational attraction of a spheroid D. Brouwer [Astr. J. 51, 223–231 (1946); MR 7, 340] obtained somewhat similar results. However, Dubošin as well as Brouwer treated only almost circular trajectories which intersect the equatorial plane of the spheroid under a small angle.

With reference to the problem stated at the beginning, the author is concerned with almost circular trajectories which intersect the plane of the equator of the spheroid at an arbitrary angle. The method used is that of Poincaré of a small parameter α, to be defined later [cf. also Orlov, ibid. 21, 25–56 (1952)]. With respect to the fixed coordinate system $Oxyz$, whose origin is at the center of the spheroid, the Oxy-plane coincides with the plane of the equator of the spheroid and the Oz-axis is oriented in such a way that the trihedral $Oxyz$ is a right-hand one, the coordinates x, y, z of the particle are given in the form of infinite series which depend upon four arbitrary parameters α, I, Ω_0 and t_0. The parameter α characterises the dimensions of the orbit of the particle. If all the terms with positive powers of α are neglected, the circular Kepler orbit of the particle with radius $1/\alpha$ is obtained. I denotes the inclination of the orbit of the particle with respect to the plane of the equator of the spheroid. The angle Ω_0 is the longitude of the node of the orbit at the initial instant t_0. Finally t_0 denotes the instant when the particle crosses the plane of the equator of the spheroid.

This theory is applicable to the study of the motion of a satellite of a planet provided that the eccentricity of the osculating orbit of the satellite is negligible and the influence of other planets and the sun is small in comparison with the influence caused by the spheroidal form of the planet under consideration. *E. Leimanis* (Vancouver, B. C.).

Orlov, A. A. On almost periodic motions of a material point in the field of gravitation of a spheroid. Moskov. Gos. Univ. Trudy Gos. Astr. Inst. 24, 139–153 (1954). (Russian)

This paper is a continuation of the paper reviewed above in which almost circular orbits of a particle under the Newtonian attraction of a nonhomogeneous spheroid of rotation were studied. In the paper under review orbits which reduce to Keplerian ellipses, if the oblateness of the spheroid is zero, are investigated.

The author is guided by the observed fact that for satellites of a planet whose orbits are subject to perturbations due mainly to the oblateness of the planet rather than to the actions of other disturbing planets, the osculating semimajor axes, eccentricities and inclinations to the plane of the equator of the planet seem to be free of secular perturbations. Accordingly, the author constructs by the small-parameter method of Poincaré solutions for the motion of a particle in the field of gravitation of a spheroid which contain only purely trigonometric terms, the unperturbed true anomaly of the particle being the independent variable.

A novel feature of the paper is that the eccentricity and the inclination of the osculating orbit of the particle are finite instead of being negligible. This assumption, however, increases the technical difficulties in computing the coordinates of the satellite to such an extent that the author has limited himself to the first approximations only. Convergence of the series involved, whose first terms only have been calculated, is not considered, nor is a proof given that the coefficients of the various powers of the parameter, according to which the expansions are carried out, can be determined as periodic functions. These questions are left for a forthcoming paper by the author. *E. Leimanis.*

Meffroy, Jean. Sur les termes séculaires du développement des grands axes par rapport aux masses. C. R. Acad. Sci. Paris 240, 1054–1056 (1955).

In this note an outline is given of the author's proof that there is a pure secular term in the development of the major axes with respect to the masses. This term is found in the third-order perturbations if the eccentricities, the inclinations and the ratio of the major axes is sufficiently small.

A. J. J. van Woerkom.

Merman, G. A. On sufficient conditions for capture in the three-body problem. Dokl. Akad. Nauk SSSR (N.S.) 99, 925–927 (1954). (Russian)

In a series of papers O. Yu. Šmidt [same Dokl. (N.S.) 58, 213–216 (1947)], O. A. Sizova [ibid. 86, 485–488 (1952); MR 14, 589], K. A. Sitnikov [Mat. Sb. N.S. 32(74), 693–705 (1953); MR 15, 356], G. A. Merman [Byull. Inst. Teoret. Astr. 5, 373–391 (1953); MR 16, 293], G. E. Hrapovickaya [ibid. 5, 435–444 (1953); MR 16, 294] constructed examples of capture in the restricted and the general three-body problem, making use of numerical integration as well as of the criteria established by Hil'mi [The problem of n bodies in celestial mechanics and cosmogony, Izdat. Akad. Nauk SSSR, Moscow, 1951] and Merman [Byull. Inst. Teoret. Astr. 5, 325–372 (1953); MR 16, 293]. In addition, Merman [last paper cited] succeeded in giving sufficient conditions for capture in the restricted hyperbolic problem of three bodies. In the present paper these results are extended to the general three-body problem. The author claims that in the example given by Šmidt (cited above) his sufficient conditions for a capture are satisfied.

E. Leimanis (Vancouver, B. C.).

Földes, István. Über die kosmogonische Theorie von O. J. Schmidt. Mat. Lapok 3, 221–236 (1952). (Hungarian. Russian and German summaries)

Es werden die Arbeiten von G. F. Hilmi über stabile und halbstabile Bewegungen im n-Körperproblem im Zusammenhange mit der Frage der Möglichkeit des Einfanges besprochen. *Author's summary.*

Kurth, Rudolf. General theory of spherical self-gravitating star systems in a steady state. Astr. Nachr. 282, 97–106 (1955).

In a spherically symmetrical distribution of gravitating mass points the equations of motion of a single particle allow the energy (E) and the angular-momentum (T) integrals

$$E=T^2+R^2+W(r)=\text{constant},\quad\text{and}\quad F=r^2T^2=\text{constant},$$

where R and T denote the radial and the transverse velocities, respectively and $W(r)$ is the gravitational potential. From Liouville's theorem it follows that the distribution function $\Psi(R, T, r)$ must be a function of E and F only. Let $\Psi(R, T, r)=f(E, F)$. The density function $D(r)$ is related

to $f(E, F)$ by

$$D(r)=\int\int f(E, F)TdRdT,$$

where the integration is extended over all values of R and T which occur. The density function $D(r)$ is related to $W(r)$ through the Poisson's equation; and the total mass of the configuration is $\int_0^\infty r^2D(r)dr$. With these definitions the following two theorems are proved.

Theorem 1: If a model possesses a finite total mass, its frequency function $f(E, F)$ has the form

$$f(E, F)=(A-E)g(E, F).$$

Here A denotes a positive constant; $g(E, F)$ is a function (defined for all $E\geqq 0$, $F\geqq 0$, non-negative, bounded, possessing continuous derivatives of the first order, if $E\neq A$) which satisfies the condition

$$g(E, F)=0 \quad \text{for all} \quad E\geqq A \quad \text{and} \quad F\geqq 0.$$

Theorem 2: Each frequency function $f(E, F)$ which vanishes identically for sufficiently large values of r defines one and only one model. *S. Chandrasekhar.*

Kurth, Rudolf. Gibt es eine statistische Mechanik der Sternsysteme? Z. Angew. Math. Phys. 6, 115–125 (1955).

The author answers in the negative the question asked in the title. His reason, stated broadly, is this: For a statistical mechanics to be possible there should exist a first integral of the equations of motion which is positive definite and which can be normalized either in the whole of the phase space or at least in an invariant subspace. The author believes that such an integral does not exist and that therefore a statistical mechanics is impossible. The paper criticizes severely all existing efforts even including those in which the time of relaxation is estimated from an analysis of two-body stellar encounters. [The reviewer cannot agree with many of the statements made in the paper and the unwary reader should be warned against the author's many categorical assertions.] *S. Chandrasekhar* (Williams Bay, Wis.).

Woolley, R. v. d. R. Relaxation of stellar velocities. Monthly Not. Roy. Astr. Soc. **114** (1954), 514–523 (1955).

The author verifies by direct calculation that in a Maxwellian distribution the number of occasions in which stars with initial velocities v_1 and v_2 encounter and end with velocities v_1' and v_2' is exactly equal to the number of occasions in which stars with initial velocities v_1' and v_2' encounter and end with velocities v_1 and v_2, i.e., we have detailed balancing. If however, the distribution of velocities is such that it is Maxwellian for $v\leqq v^*$ (say) and stars with velocities exceeding v^* do not exist, then detailed balancing no longer holds and one can ask how long it will take for the "tail" of the distribution to be replaced. The author shows that this time is very long compared with the time of relaxation as usually defined so long as v^* is appreciably greater than about twice the root-mean-square velocity. From this result the author concludes that the use of a "truncated" Maxwell distribution of velocities may be justified in some contexts of stellar dynamics.

S. Chandrasekhar (Williams Bay, Wis.).

Lundquist, Charles A., and Horak, Henry G. The transfer of radiation by an emitting atmosphere. IV. Astrophys. J. **121**, 175–182 (1955).

The present investigation is concerned with the solution of the equations of radiative transfer in plane-parallel atmospheres of finite optical depth τ, in which the source function ϵ for the emission sources depends on τ alone, and light is scattered isotropically in accordance with an albedo coefficient $\tilde{\omega}_0$. The equations of the problem then assume the form

$$\mu\frac{dI(\tau, \mu)}{d\tau}=I(\tau, \mu)-\tilde{\omega}_0J(\tau)-\epsilon(\tau),$$

$$J(\tau)=\frac{1}{2}\int_{-1}^{1} I(\tau, \mu)d\mu,$$

where $I(\tau, \mu)$ denotes the intensity of radiation at an optical depth τ, and μ is cosine of the angle between the incident and scattered light.

The solution of a general problem of this type is shown to be expressible as the sum of the solutions of two other problems: one having specified boundary conditions and no emission sources; the other characterized by given distribution of sources and no incident light. The complete solution of the first problem was given by Chandrasekhar [Radiative transfer, Oxford, 1950; MR **13**, 136]; while the second was solved for a uniform distribution of sources by Horak [Astrophys. J. **116**, 477–490 (1952); MR **14**, 804], and for a linear distribution of energy sources by Horak and Lundquist [ibid. **119**, 42–50 (1954); MR **15**, 750]. In the present paper, its authors are concerned with a construction of solutions for the case when $\epsilon(\tau)$ is an arbitrary polynomial of its argument. Part II of their investigation deals with semi-infinite atmospheres; parts III and IV with non-conservative and conservative scattering in finite atmospheres, respectively. In part V the authors consider $\epsilon(\tau)$ to be a quadratic function of τ, and in part VI construct the corresponding numerical solution by the method of discrete ordinates. *Z. Kopal* (Manchester).

Lundquist, Charles A. The transfer of radiation by an emitting atmosphere. V. Astrophys. J. **121**, 183–189 (1955).

The present investigation considers the problem of radiative transfer in finite and semi-infinite plane-parallel atmospheres scattering isotropically and emitting in accordance with source functions ϵ of the form $\epsilon(\mu)=\mu^n$ and $\epsilon(\tau, \mu)=\mu^n\tau^m$, where μ is cosine of the angle between incident and scattered light; τ, the optical depth; and m, n are positive integers. The paper is an extension of the earlier work by Horak and Lundquist [Astrophys. J. **119**, 42–50 (1954); MR **15**, 750; see also the paper reviewed above] in which source functions $\epsilon(\tau)$ alone were considered, and employs the method of the 1954 investigation. In part II of the present paper, the author considers the case of finite atmospheres with non-conservative scattering and $\epsilon(\mu)=\mu^n$; in parts III and IV he treats the case of conservative scattering in finite and semi-infinite layers characterized by the same source function; and part V contains a brief discussion of the more general case of a source function $\epsilon(\tau, \mu)=\tau^m\mu^n$. *Z. Kopal.*

King, Jean I. F. Radiative equilibrium of a line-absorbing atmosphere. I. Astrophys. J. **121**, 711–719 (1955).

The radiative equilibrium of a non-gray atmosphere is investigated in the special case when the variation in the absorption coefficient corresponds to equally spaced, equally intense, Lorentz-broadened absorption lines. It is shown that in this case the methods which have been developed [S. Chandrasekhar, Radiative transfer, Oxford, 1950, chaps. iii, iv; MR **13**, 136] for solving the corresponding gray problem can be successfully extended. In particular, the method of

replacing the integro-differential equation by an equivalent system of linear equations and the method based on the principles of invariance both apply. Thus the exact solution for the angular distribution of the emergent radiation can be expressed in terms of H-functions suitably defined. Convenient tables are provided for obtaining the explicit form of the solutions in practical cases. *S. Chandrasekhar.*

Bouvier, Pierre. Une extension particulière de la méthode de Wiener-Hopf. Arch. Sci. Soc. Phys. Hist. Nat. Genève **8**, 87–93 (1955).

The known solution [S. Chandrasekhar, Radiative transfer, Oxford, 1950, see §§11.2, 45, and chap. 5; MR **13**, 136] for the problem of radiative transfer in a semi-infinite plane parallel atmosphere scattering radiation in accordance with Rayleigh's phase function is obtained by a method similar to that of Wiener-Hopf for the case of isotropic scattering.
S. Chandrasekhar (Williams Bay, Wis.).

Münch, Guido. The theory of the fluctuations in brightness of the milky way. VI. Astrophys. J. **121**, 291–299 (1955).

The author of the present paper continues the previous investigations of stochastic processes defined by the surface brightness of the Milky Way on the discrete cloud model [cf. Chandrasekhar and Münch, Astrophys. J. **112**, 380–392, 393–398 (1950); **114**, 110–122 (1951); MR **12**, 644; **13**, 249]. The investigation under review presents a solution of the problem of finding the probability distribution for the joint occurrence of two specified values of the surface brightness when we observe parts of the system extending to two given distances. This solution is obtained in terms of a set of functions $\{\Phi_n(u,t)\}$ governed by the differential equation

$$\Phi_n(u,t)+\frac{t}{n}\left\{\frac{\partial\Phi_n}{\partial u}+\frac{\partial\Phi_n}{\partial t}\right\}=\Phi_{n-1}\left(\frac{u}{q},t\right),$$

which gives the probability distribution of the brightness u, conditioned to the occurrence of exactly n clouds in a system of extent t. The covariance of the process is shown to be of a simple exponential type, with a decay constant equal to $(1-q)^{-1}$, q being the transparency factor of a cloud. If we adopt values of q and ν (the mean number of clouds per unit distance) previously found, it is estimated that the distance over which the brightness fluctuations become effectively uncorrelated is approximately 600 parsecs. Accordingly we should expect, for instance, that the color excesses of the B-stars should be uncorrelated with the local values of the surface brightness of the Milky Way, as seems indeed to be borne out by the observations. *Z. Kopal.*

Kippenhahn, Rudolf. Zur Dynamik eines von Strahlung durchsetzten Mediums. Z. Astrophys. **35**, 165–178 (1954).

The author of the present investigation points out that, in contrast to a medium at rest, the tensor expression for the radiative pressure in a moving medium contains terms which are bound to produce energy exchange between radiation and moving matter. Consequently, the Eulerian hydrodynamical equations of motion must be augmented by terms arising from this source, whose magnitude is likely to make them of importance in stellar interiors. The Jeans's theory of the "braking effect" of radiation [Monthly Not. Roy. Astr. Soc. **86**, 328–335 (1926)] in rotating stars proves to be a special case of the more general theory outlined in the present paper. *Z. Kopal* (Manchester).

Kaplan, S. A. The distribution function of the velocities of turbulent motion of an interstellar gas. Astr. Ž. **31**, 137–140 (1954). (Russian)

The author derives a velocity distribution function for turbulent flow by a method alternative to that by Huang [Astrophys. J. **112**, 399–417 (1950); MR **12**, 545]; and employs the observational data on radial velocities of interstellar clouds [cf. Adams, ibid. **109**, 354–379 (1949)] to construct an empirical velocity-distribution, which is found to be in good agreement with the theory. *Z. Kopal.*

Kaplan, S. A. Isothermal flow of a gas in interstellar space. Discontinuities in density and velocity. Astr. Ž. **31**, 31–35 (1954). (Russian)

The author's aim is to analyze the distribution of velocity and density in plane gas flows which may arise in interstellar space. In the first part of the paper under review the equations of one-dimensional gas flow are set up, and approximate solutions obtained in the form of series expansions in ascending powers of the parameter $w=\log(\rho/\rho_0)$, where ρ denotes the instantaneous density at any point of the actual gas flow, and ρ_0 the density of the undisturbed medium. In the second part of his paper, the author considers the possibility of the formation of plane shocks in such a flow, and determines the corresponding density and velocity jumps across the shock front. *Z. Kopal* (Manchester).

Sweet, P. A. Field reversal in magnetic variable stars. Monthly Not. Roy. Astr. Soc. **114** (1954), 549–557 (1955).

As has been shown by Cowling [same Not. **112**, 527–539 (1953)] in the oscillation theories of magnetic variable stars (where the magnetic field variation is ascribed to distortion in the lines of force by moving material in the star), the field in the reversing layer should be equal to the surface value of the field outside the star. As this latter is a potential field, a condition is thus imposed on the reversing-layer field, limiting the possible variation in the resultant field strength derived from the longitudinal Zeeman effect in the star's spectrum. The author of the paper under review set out to examine this condition more fully. His discussion is restricted to fields which remain symmetrical about a fixed axis during their variation. For spherical stars (darkened by 45% at limb) it is found that (provided suitable gas motions can arise) the reversal is possible only in the following cases: (i) dipole-type fields, with distortions that are not symmetrical about the equatorial plane, when the inclination of the magnetic axis to the line of sight exceeds 79°; (ii) symmetrical quadrupole-type fields with unsymmetrical distortions, when this inclination is between 50° and 69°; (iii) quadrupole-type fields with unsymmetrical distortions, for all inclinations; and lastly (iv) certain fields of higher complexity even when symmetrical, for all inclinations. In conclusion the author points out, however, that (with the exception of the non-symmetrical quadrupole types) it is unlikely on statistical grounds that the fields considered could account for the observed facts, and suggests that more complex (or non-axially symmetric) fields will be required to reconcile the oscillation theories with observation.
Z. Kopal (Manchester).

Dungey, J. W. Deductions from the perfect cosmological principle. Proc. Cambridge Philos. Soc. **51**, 532–535 (1955).

The author derives some of the formulae describing the steady-state world-model of Bondi and Gold from the perfect cosmological principle (p.c.p.) alone, without appeal to

relativity. The method used depends on a convention for the measurement of distance which implies no contraction in the apparent length of a moving rod. It also depends on the sign of the universal constant T in the shift-distance law, derived by the author, $\nu_Q/\nu_P = \exp(-a_{PQ}/T)$, where ν_P, ν_Q denote the respective frequencies of emission and absorption and a_{PQ} is the distance PQ. The author fails to show, however, why T must be positive, which is essential for red-shifts. In fact, the p.c.p. is not powerful enough for this purpose, since it tells us nothing about the direction of time. The principle could equally well give rise to a static model or to one associated with blue-shifts and continued annihilation of matter. *G. J. Whitrow* (London).

McVittie, G. C. Relativity and the statistical theory of the distribution of galaxies. Astr. J. 60, 105–115 (1955).

J. Neyman and E. L. Scott [Astrophys. J. 116, 144–163 (1952); MR 14, 803] have constructed a theoretical model for the spatial distribution of galaxies and clusters of galaxies, and, together with C. D. Shane [ibid. 117, 92–133 (1953)], have studied the connection between this model and certain observational data. The Neyman-Scott theory takes account of the red shift only insofar as it affects the apparent magnitudes of external galaxies. The author attempts to modify the theory so as to take fuller account of the expansion, as well as the possible curvature, of space. To this end he makes certain changes in the interpretation of geometrical quantities that enter into the formulae of Neyman and Scott, replacing the Euclidean distance between the observer and an external galaxy by the 'luminosity distance' [W. H. McCrea, Z. Astrophys. 9, 290–314 (1935)], the Euclidean volume element by the proper volume element in a spatially isotropic expanding universe, and so on. In addition, he reformulates the Neyman-Scott postulate of statistical uniformity along the following lines.

Let dV_1 and dV_2 be non-overlapping proper volume elements around two points, P_1 and P_2, which lie on the light cone of an observer O at some instant t_0. The proper volume element dV_i is given by

$$dV_i = R^3(t_i)\frac{r_i^2 dr_i \sin\theta_i d\theta_i d\varphi_i}{(1+\frac{1}{4}kr_i^2)^3}.$$

Then the probability of finding n clusters in dV_1 is independent of the number found in dV_2, and is equal to the probability of finding n clusters in dV_2 if $dV_1 = dV_2$.

In the reviewer's opinion the author's uniformity postulate is not consistent with his use of a uniform and isotropic world model, unless he makes the additional assumption that matter is being created at such a rate that the proper density of matter remains constant in time. If matter is conserved, the probability of finding n particles in dV_1 is equal to the probability of finding n particles in dV_2 if the co-ordinate volumes $dV_1/R^3(t_1)$ and $dV_2/R^3(t_2)$ are equal [McCrea, loc. cit.]. In a non-static universe $R(t_1) = R(t_2)$ if and only if the points P_1 and P_2 are equidistant from the observer. *D. Layzer* (Cambridge, Mass.).

Brahmachary, R. L. On the cosmological implication of galactic magnetic fields. Nuovo Cimento (10) 1, 953–954 (1955).

RELATIVITY

✶Gomes, Ruy Luís. A teoria da relatividade. Espaço-tempo-gravitação. [The theory of relativity. Space-time-gravitation.] Edições Monsanto, Lisbon, 1954. vii+86 pp.

A series of lectures on the special and general relativity theories, emphasizing geometrical and kinematical aspects almost to the exclusion of physical content. Thus the problem of setting up and integrating Einstein's gravitations is left completely untouched. The book is in large part a protest against the extreme position, attributed to followers of Minkowski, which holds space-time to be an absolute and indivisible whole. Gomes prefers instead to exploit the separation into proper times and spaces relative to material world-lines. In this he associates himself with the position taken by C. Møller in his "The theory of relativity" [Oxford, 1952; MR 14, 212], but goes beyond it to consider the conditions under which space-time may be broken down into what he calls "absolute" space and time.

Chapter I presents concisely the principles of classical mechanics, with particular attention to conceptual foundations. A brief Chapter II considers the motion of light and the derivation therefrom of the Lorentz transformations. The special theory of relativity is next taken up in Chapter III, with emphasis on the structure of Minkowski space-time. The equations of motion are formulated, and the former applied to Born's hyperbolic motion.

The principal thesis of the book is treated in the remaining 30 pages—Chapter IV, The general theory of relativity. Here the emphasis is on the breaking down of space-time into proper spaces and times, relative to the world-lines of matter. Following Møller, this is initiated by constructing a 3-dimensional line element which serves as the metric of a local proper space in the neighborhood of an event on a given world line, to which it is clearly relative. Gomes then considers the conditions under which the spatial elements associated with events on a congruence of time-like world-lines may be united into a single space imbedded in physical space-time. When these apply, he considers the resolution as giving "absolute" space and time, even although the process is admittedly relative to the given normal congruence of world-lines—as in the usual relativistic cosmological models. The contrary situation is illustrated by a rotating coordinate frame in special relativity, and by K. Gödel's remarkable cosmological model in which all matter is in rotation. *H. P. Robertson* (Paris).

Gomes, Ruy Luís. "Absolute" space and time in classical physics and in the theory of relativity. Ciência 4, nos. 9–10, 13–27 (1954). (Portuguese)

A concise treatment of the problem of significantly breaking down space-time into space and time, relative to the material content. Gomes treats the subject more fully in the book reviewed above in which his position is developed about this point. *H. P. Robertson* (Paris).

Clauser, Emilio. Sui fronti d'onda nella teoria unitaria einsteiniana. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 18(87), 473–492 (1954).

The author seeks the characteristic hypersurfaces of the unified field-equations in the case of a particular kind of space-time introduced by Hlavatý [J. Rational Mech. Anal. 2, 1–52 (1953); MR 14, 305], in which the null autoparallels coincide with the null geodesics of the associated Riemannian space-time. The author's problem is related to, but

is not the same as, the Cauchy problem treated by Lichnerowicz [ibid. 3, 487–521 (1954); MR 16, 408]. His conclusion is that the equation of the characteristic hypersurfaces in the associated Riemannian space-time is the same as that of general relativity, and that a similar conclusion holds for a class of space-times analogous to those of Hlavatý. Thus the action of the unitary field is propagated with the velocity of light, like that of the gravitational field in general relativity. *H. S. Ruse* (Leeds).

Graiff, Franca. Formule di commutazione e trasporto ciclico nei recenti spazi di Einstein. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18**(87), 105–110 (1954).

Associated with the asymmetric connection of the Einstein unified theory are the two kinds of covariant derivative, which, in the case of a covariant vector A_i, are given by

$$A_{\underset{+}{i;k}}=\partial A_i/\partial x^k-A_s\Gamma^s{}_{ik},$$

$$A_{\underset{-}{i;k}}=\partial A_i/\partial x^k-A_s\Gamma^s{}_{ki}.$$

There are thus various second-order derivatives according as the existing indices are treated as of "plus" or "minus" type for the second differentiation, and hence a number of permutation-formulae, of which the simplest are

$$(*)\qquad A_{\underset{\underset{+}{++}}{i;kr}}-A_{\underset{\underset{+}{+-}}{i;rk}}=-A_sR^s{}_{ikr}$$

and the formula obtained from this by replacing the connection by its conjugate. Other formulae (for example the one obtained by taking plus-derivatives throughout) involve the torsion-tensor as well as the curvature-tensor on the right-hand side. The author's purpose in the present paper is to show that the existence of the simple formula (*) is related to the possibility of constructing a closed infinitesimal quadrangle at any point P by transporting vectors dP and δP at P respectively by plus- and minus-parallelism along δP and dP. [See Graiff, Bol. Un. Mat. Ital. (3) 7, 132–135 (1952); MR 14, 316.] *H. S. Ruse* (Leeds).

Hennequin, Françoise. Interprétation de la théorie de Y. Thiry dans une métrique conforme. C. R. Acad. Sci. Paris **240**, 2378–2380 (1955).

In the unified field theory of Y. Thiry [J. Math. Pures Appl. (9) **30**, 275–316, 317–396 (1951); C. R. Acad. Sci. Paris **235**, 1480–1482 (1952); MR 13, 787; 14, 591] there is a V_5 with metric $d\sigma^2=\gamma_{\alpha\beta}dx^\alpha dx^\beta$ ($\alpha, \beta=0, 1, 2, 3, 4$), the coefficients being independent of x^0, and a V_4 (space-time) with metric $ds^2=g_{ij}dx^idx^j$ ($i, j=1, 2, 3, 4$), where

$$g_{ij}=\gamma_{ij}-\gamma_{0i}\gamma_{0j}/\gamma_{00}.$$

The author proposes a modification of Thiry's theory, taking as space-time V_4^* with $ds^{*2}=g^*{}_{ij}dx^idx^j$, $g^*{}_{ij}=\xi g_{ij}$, $\xi^2=-\gamma_{00}$. The trajectory of a charged particle is then an extremal of

$$\int[(\alpha^2/\xi^2-1)^{1/2}\xi^{-1/2}ds^*+\alpha\beta\phi_idx^i],$$

where ϕ_i is a 4-potential defined by $\gamma_{0i}=\beta\phi_i\gamma_{00}$ and α, β are constants. This modification simplifies the form of approximate solutions of the field equations [ibid. **239**, 1464–1466 (1954); MR 16, 872]. It also ensures that families of isothermal varieties in V_5 project into families of isothermal varieties in V_4^*. *J. L. Synge* (Dublin).

Dewitt, Cécile Morette, et Dewitt, Bryce S. Sur une théorie unitaire à cinq dimensions. I. Lagrangien. C. R. Acad. Sci. Paris **241**, 167–168 (1955).

Zanella, Angelo. Successive linearizzazioni in una recente teoria relativistica unitaria. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18**(87), 575–592 (1954).

The author takes a set of equations for a unified field-theory proposed by Finzi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 581–588 (1953); MR 15, 563]. These, in a notation now sufficiently well known not to need detailed specification, are

$$g_{\underset{+-}{ik;r}}=0;\qquad R^*{}_{\underline{ik}}=0;$$

$$\operatorname{curl} R^*{}_{\underset{\smile}{ik}}=0;\quad \operatorname{div} R^*{}_{\underset{\smile}{ik}}=0,$$

where

$$R^*{}_{ik}=R_{ik}+\Gamma_{\underset{+}{i;k}},$$

R_{ik} being the contracted curvature-tensor and Γ_i the contracted torsion-tensor. In this theory $R^*{}_{\underset{\smile}{ik}}$ is identified with the electromagnetic tensor. The author examines these equations by successive approximations. He first takes

$$g_{ik}=a_{ik}+b_{ik},$$

where the a_{ik} (symmetric) have galilean values, and the b_{ik} (asymmetric) are small compared with a_{ik}. Later he proceeds to a second approximation by taking

$$g_{ik}=a_{ik}+b_{ik}+c_{ik},$$

where the c_{ik} are small of the second order, and concludes with a discussion of the corresponding approximation of the nth order. [Cf. Udeschini, ibid. **9**, 256–261 (1950); **10**, 21–24, 121–123 (1951); MR 12, 864; 13, 169.] For the first approximation the author obtains, in a particular reference-system,

$$\Delta b_{\underline{ik}}=0,\quad \operatorname{curl}\Delta b_{\underset{\smile}{ik}}=0,\quad \operatorname{div}\Delta b_{\underset{\smile}{ik}}=0,$$

where Δ denotes the dalembertian, and notes how the gravitational and electromagnetic effects separate out. The second approximation, however, leads to equations of which the first, namely

$$\Delta c_{\underline{ik}}=B_{\underline{ik}},$$

is typical, the quantities B_{ik} depending on both $b_{\underline{ik}}$ and $b_{\underset{\smile}{ik}}$. Thus in this and in higher approximations gravitational and electromagnetic effects are inseparable. The paper includes the discussion of a particular case that illustrates the first approximation. *H. S. Ruse* (Leeds).

Schücking, E. Das Schwarzschildsche Linienelement und die Expansion des Weltalls. Z. Physik **137**, 595–603 (1954).

The author seeks, in continuation of the investigations of Einstein and Straus [Rev. Mod. Phys. **17**, 120–124 (1945); MR 7, 87], the relativistic gravitational field of a massive particle imbedded in an expanding universe. In suitably chosen coordinates, a hole of radius r_0 is cut in an expanding model, and the Schwarzschild vacuum field generated by a particle of mass M is fitted within the hole in such a way that the gravitational potentials are continuous at the boundary; the composite solution is here obtained in finite terms. (However, the author does not examine whether the vacuole fits smoothly into the exterior solution at the boundary; discontinuities in the normal derivatives of the gravitational potentials imply a surface distribution of matter required to maintain the exterior solution.) The

radius r_0 of the vacuole is found, in agreement with Einstein and Straus, to be proportional to the cube root of the ratio M/ρ_M, where ρ_M is the mean density of matter in the expanding exterior. (The mass removed is, to first approximation, just equal to that of the particle.) From this the author concludes that the local galactic group is within its own vacuole, but that of the Virgo cluster engulfs the local group. *H. P. Robertson* (Paris).

Pirani, F. A. E. On the perihelion motion according to Littlewood's equations. Proc. Cambridge Philos. Soc. **51**, 535–537 (1955).

D. E. Littlewood's suggestion [same Proc. **49**, 90–96 (1953); MR **14**, 572] that Einstein's equations of the gravitational field in the absence of matter be replaced by the equation $R=0$, where R is the curvature scalar $(R=R_\mu^\mu)$, together with Littlewood's assumption that space-time is conformally flat is shown to lead to a planetary perihelion motion of one-sixth the amount derived from the Schwarzschild solution in Einstein's theory and of the opposite sign. *G. J. Whitrow* (London).

Takasu, Tsurusaburo. Equations of motion of a free particle in the author's general relativity as a non-holonomic Laguerre geometry realized in the moving three-dimensional Cartesian space. Proc. Japan Acad. **30**, 814–819 (1954).

The author shows that the values obtained by means of his new theory of relativity [same Proc. **30**, 702–708 (1954); MR **16**, 1058] for the advance of perihelion, the deflection of light and the displacement of the Fraunhofer Lines coincide with Einstein's results. *J. Haantjes* (Leiden).

McVittie, G. C. Gravitational waves and one-dimensional Einsteinian gas-dynamics. J. Rational Mech. Anal. **4**, 201–220 (1955).

Here gravitational waves are said to exist when the field equations in empty space-time are shown to have solutions $g_{\mu\nu}$ which are time-dependent and which are also solutions of the wave equation. [This definition is not invariant and does not seem to have any physical significance, since the author has repeatedly to reject metrics satisfying these conditions which are either flat or can be transformed into time-independent form or both, and so do not represent "genuine gravitational wave solutions".] He approaches the problem by seeking "ultimately empty" space-times, using the technique developed in a previous paper [Quart. Appl. Math. **11**, 327–336 (1953); MR **15**, 175] but finds no gravitational waves which are physically acceptable to him. *F. A. E. Pirani* (London).

McVittie, G. C. Relativistic and Newtonian cosmology. Astr. J. **59**, 173–180 (1954).

The author's purpose is to show how, and to what extent, Newtonian cosmological models are derivable from those of the general theory of relativity. To this end he sets up the general spherically symmetric time-dependent field in general relativity. By appropriate specialization he obtains therefrom the well-known spatially uniform cosmological models. Returning to the general field he obtains, by limiting processes defining the transition from relativistic to Newtonian theory, models which he interprets classically. The author finds as the only Newtonian analogues ones in which the time dependence is in fact the same as in those relativistic models for which the cosmological constant (and the pressure) vanish, although a vestige of this constant remains as an additive constant in the pressure. He concludes that the cosmological constant does not manifest itself by a repulsive force proportional to the distance in the Newtonian approximation, as is generally held. *H. P. Robertson* (Paris).

Havlíček, Franjo I. On the problem of a variable gravitational constant. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **276**, 101–107 (1949). (Serbo-Croatian)

An attempt to found a non-static cosmological model by assuming the red-shift of spectral lines in light as an effect of strong gravitational fields. To justify this hypothesis the author supposes the dependence of the gravitational constant f from time to be in the form $f\approx 1/\rho T^2$, where ρ is the constant density of the universe and T the time from the very instant of the generation of a given part of the universe. As a cosmological principle he assumes that the universe is expanding from a nucleus with the velocity of light during which the masses are materialized in the average density of the universe. He analyzes principally the influence of this variable gravitational constant on the Friedman's equations [Z. Physik **10**, 377–386 (1922) = Ž. Fiz. Him. Obšč. **56**, 59–68 (1924) (in Russian); cf. also Robertson, Proc. Nat. Acad. Sci. U. S. A. **15**, 822–829 (1929); R. C. Tolman, Relativity, thermodynamics and cosmology, Oxford, 1934, p. 361 ff.]. *T. P. Anđelić* (Belgrade).

Havlíček, Franjo I. Zur Frage der veränderlichen Gravitationskonstante. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **5**, 33–35 (1952).

Summary of the paper reviewed above.

Loedel, Enrique. Direct deduction of the three crucial effects of Einstein's theory of gravitation starting from the principle of parabolic velocity. Actas Acad. Ci. Lima **17**, 3–38 (1954). (Spanish)

The author asks whether it is possible to obtain a more immediate understanding of the solar gravitational field than by arriving at the Schwarzschild line element via Einstein's field equations. The answer is sought in terms of a modified principle of equivalence, which states that a static gravitational field in the neighborhood of a given point is equivalent to motion in a moving system K' whose velocity is the parabolic velocity at the point in question. The author derives the implications of this principle for motion in the solar field, in which a leading role is played by the Fitz-Gerald-Lorentz contraction factor—given for parabolic velocity by the square root of $1-2GM/c^2r$. Finally he applies the contraction and local time factors directly to the Minkowski line element of the moving system K', to obtain the Schwarzschild metric in the static coordinate system. [There may be some connection between this approach and that of Kirkwood [Phys. Rev. (2) **92**, 1557–1562 (1953); MR **15**, 657] who takes as fundamental reference system an ether moving with parabolic velocity.] *H. P. Robertson* (Paris).

Kirkwood, Robert L. Gravitational field equations. Phys. Rev. (2) **95**, 1051–1056 (1954).

Guided by the postulate that "the motion or the magnitude of any physical quantity that is located within a given volume in space is not directly influenced by the motion or the magnitude of any other quantity that is located outside of this volume, regardless of how small the volume may be", the author presents cogent arguments for an all-pervading

ether as the carrier of influence. This differs radically from the classical fixed ether theories in that here the ether is in motion, and the effects of this motion are realized as gravitation. He defines carefully the types of physical quantities which are allowed by his postulate to enter into the theory, and concludes in particular that only the strain tensor and the second and higher derivatives of the ether velocity v may appear, in addition of course to the physical quantities specifying the particular system under consideration. By comparing his equations of motion of a particle, as derived in a previous article [Phys. Rev. (2) 92, 1557–1562 (1953); MR 15, 657], with the Newtonian gravitational equations, he obtains the equations of motion of the ether; the argument could be reversed to say that, starting with these equations of motion of the ether, one obtains automatically a classical theory of gravitation.

The author points out the connection of this work with his previous paper, in which by allowing the mass of a particle to vary in the manner suggested by the special theory of relativity, he obtained equations of motion for the Kepler problem which were in fact identical with those obtained from the Schwarzschild field in the general theory of relativity. *H. P. Robertson* (Paris).

Bondi, H., and Gold, T. The field of a uniformly accelerated charge, with special reference to the problem of gravitational acceleration. Proc. Roy. Soc. London. Ser. A. **229**, 416–424 (1955).

The authors criticise previous derivations of the field of a uniformly accelerated charge [M. Born, Ann. Physik (4) **30**, 1–56 (1909), p. 39; this field does not vanish outside the envelope E of light-cones drawn from the charge; G. A. Schott, Electromagnetic radiation, Cambridge, 1912, pp. 63–69; this is the Born field cut off at E, but on E it violates Maxwell's equations]. They attribute the failure of the method of retarded potentials in deriving the Born field to the divergence of the contribution to the field from distant portions of the world-line of the charge.

Using two physically acceptable limiting processes which both lead to the same result, they rederive the field, arriving at a result which is the same as Schott's field with the addition of δ-function terms on E. They conclude that a uniformly accelerated charge radiates, but not a charge statically supported against gravitation. This need not contradict the principle of equivalence. *F. A. E. Pirani.*

Meister, H. J., und Papapetrou, A. Die Bewegungsgleichungen in der allgemeinen Relativitätstheorie und die Koordinatenbedingung. Bull. Acad. Polon. Sci. Cl. III. **3**, 163–168 (1955).

The second author's calculations of the equations of motion [Proc. Phys. Soc. Sect. A. **64**, 302–310 (1951); MR **13**, 695] are summarized and it is shown in addition that for spherically symmetrical particles equations of motion in the first approximation identical with those found by the EIH method [latest and most elegant derivation: L. Infeld, Acta Phys. Polon. **13**, 187–204 (1954); MR **16**, 531] may be derived. *F. A. E. Pirani* (London).

Grünbaum, Adolf. The clock paradox in the special theory of relativity. Philos. Sci. **21**, 249–253 (1954).
Leaf, Boris. The clock paradox in the special theory of relativity. Philos. Sci. **22**, 45–52 (1955).

The perennial "paradox" of the travelling twin, complicated by worrying about clock readings for the significant events in various reference systems. Sisyphus rides again!

The paradox is again satisfactorily disposed of by Grünbaum, who proves the consistency of the results obtained by the various observers. This is denied by Leaf, who incorrectly ascribes an error to Grünbaum (footnote, p. 47), and who then proceeds to make a number of errors of his own (last sentences on pp. 48, 50). These lead him to untenable conclusions (p. 51), which he considers as establishing consistency. *H. P. Robertson* (Paris).

★**Synge, J. L. Relativistically rigid surfaces.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 217–226. Academic Press Inc., New York, 1954. $9.00.

A relativistically rigid surface is here defined as one such that during its motion the world lines of neighboring points maintain a constant perpendicular distance, as measured by the Minkowski metric of flat space-time. The general kinematical equations of motion of the surface are obtained, involving the velocities of the superficial points and the parametric equations of the moving surface, and expressions for the deformations in the parametric directions are found. The form assumed by a spherical surface rotating with arbitrary angular velocity about fixed axis is rigorously discussed. The slow motion of an arbitrary surface is described in the first approximation by the 6-parameter Newtonian group, the resulting deformations are interpreted as FitzGerald-Lorentz contractions due to the component of velocity in the direction concerned, and the differential equations for the resulting displacements in this order are set up. From this the author conjectures that the rigorous motion is also described by a 6-parameter group. *H. P. Robertson.*

Pounder, J. R. On relativistically rigid surfaces of revolution. Comm. Dublin Inst. Advanced Studies. Ser. A. no. **11**, iv+53 pp. (1954).

The concept of a rigid body cannot be taken over into the special theory of relativity without severe loss of mobility, as shown by the investigations of Born [Ann. Physik (4) **30**, 1–56 (1909)] and of Herglotz [ibid. **31**, 393–415 (1910)]. To restore mobility, Synge (in the paper reviewed above) has recently advanced the notion of a superficially rigid body, in which only the surface satisfies the Born criterion of rigidity. Pounder here investigates the special case of a rigid surface of revolution in uniform rotation about or uniform screw motion along its axis. The consequent distortions of the surface, as viewed from an inertial frame, are interpreted with the aid of the Fitzgerald-Lorentz contraction. *H. P. Robertson* (Paris).

Kraichnan, Robert H. Special-relativistic derivation of generally covariant gravitation theory. Phys. Rev. (2) **98**, 1118–1122 (1955).

The author studies Lorentz-invariant field theories (for a symmetrical tensor field) which are derivable from a variational principle. By getting rid of that part of the Lagrangian which depends on the Minkowskian metric tensor he develops generally covariant equations (including those of general relativity) supplemented by noncovariant auxiliary conditions. A further paper discussing physical significance is promised. *F. A. E. Pirani* (London).

Gomes, Ruy Luis. The notion of rigid body in Restricted Relativity. Gaz. Mat., Lisboa 15, no. 58, 9–11 (1954). (Portuguese)

The author derives a precise criterion for rigid-body motion in special relativity, based on M. Born's definition [Ann. Physik (4) 30, 1–56 (1909)]. The coordinates chosen are three Lagrangian labels for the world-lines of the body, and the proper time measured along the world-lines. The condition for rigidity is that in the new coordinates the metric of the associated proper space be independent of the proper time. *H. P. Robertson* (Paris).

Kalitzin, Nikola St. Relativistische Mechanik des materiellen Punktes mit veränderlicher Masse. C. R. Acad. Bulgare Sci. 7 (1954), no. 2, 9–12 (1955). (Russian. German summary)

The equations of motion are derived within the framework of the classical special relativity theory for a particle having a variable rest-mass. *N. Rosen* (Haifa).

Marx, G., and Szamosi, G. Relativistic motion in a scalar field. Bull. Acad. Polon. Sci. Cl. III. 2 (1954), 475–479 (1955).

The classical relativistic equations of motion of a particle in a scalar field are investigated. It is found that the rest-mass is in general not constant. In the case of a scalar, attractive, meson field, the rest-mass (but not the total mass or energy) will vanish at some distance from the center of force and the velocity there will be equal to that of light. For smaller distances the effective force on the particle will be repulsive. *N. Rosen* (Haifa).

Takeno, Hyôitirô. On equivalent observers. II. Progr. Theoret. Physics 12, 129–140 (1954).

A continuation of the paper by Ueno and Takeno [same journal 8, 291–301 (1952); MR 14, 506] seeking groups of motions $\mathfrak{G}$ suitable for describing the relations between equivalent observers. Here the group $\mathfrak{G}_6$ of frame transformations, taken in I as spatial rotations and translations plus translation in time, is augmented by dilatations of the space-time, space or time coordinates. In the first case it is concluded that the only full groups must be either the Lorentz or Galilean group, with more complicated possibilities arising where the space and time dilatations may differ. The author concludes with an analysis of the problem of a 1-parameter set of "linear equivalent observers". *H. P. Robertson* (Paris).

MECHANICS

Phillips, J. R. A graphical method for skew forces and couples. Austral. J. Appl. Sci. 6, 131–148 (2 plates) (1955).

In colorful, and sometimes obscure, language the paper presents the Moebius theorem (1839) about the lines of action of four forces in equilibrium [cf. E. J. Routh, A treatise on analytical statics, v. 1, Cambridge, 1891, arts. 316–318; the paper quotes only arts. 261 and 271 of the 1896 edition], and muses upon some of its implications. There are two photographs of demonstration models for the theorem, and some graphical constructions in the paper. In these times of few readers and many writers, revivals of old forgotten, and not easily accessible matter may be sometimes welcome. *A. W. Wundheiler* (Chicago, Ill.).

Kalitzin, G. St. Die Begründung der Getriebelehre durch die Mengenlehre. Acta Tech. Acad. Sci. Hungar. 11, 441–448 (1955). (Russian, English and French summaries)

Kalitzin, G. St. Gruppentheoretische Eigenschaften der Getriebe und Anwendung der Matrizenrechnung zur Berechnung von Getrieben. Acta Tech. Acad. Sci. Hungar. 11, 449–460 (1955). (Russian, English and French summaries)

Tzénoff, Ivan. Sur une forme nouvelle des équations de la mécanique analytique et quelques applications de ces équations. Bŭlgar. Akad. Nauk. Izv. Mat. Inst. 1, no. 2, 91–134 (1954). (Bulgarian. French summary)

This is a republication of previous results [Dokl. Akad. Nauk SSSR (N.S.) 89, 21–24, 225–228, 415–418, 623–626 (1953); MR 14, 916; 15, 259]. The author uses the variable $\bar{T}$ as a function of t, q, $\dot{q}$, $\ddot{q}$, and $\dddot{q}$. It may be remarked, that if the order of the complete time-derivatives of the kinetic energy and the generalized coordinates is not restricted, infinitely many forms of the equations of analytical dynamics may be produced. *A. W. Wundheiler.*

Schieldrop, Edgar B. A principle in classical mechanics with a 'relativistic' path-element extending the principle of least action. Proc. Cambridge Philos. Soc. 51, 469–475 (1955).

The author first derives Jacobi's Principle $\delta\int(h-V)^{1/2}ds=0$ for a particle in a stationary conservative field directly from Lagrange's equations. By a slight change in the method he then finds the variational principle $\delta\int V^{1/2}(h-T)^{1/2}dt=0$ and shows that Jacobi's Principle is a special case of this but not conversely. Now letting $h=\frac{1}{2}mc^2$, he obtains the variational principle $\delta\int V^{1/2}d\sigma=0$, where $d\sigma=(c^2dt^2-dx_1^2-dx_2^2-dx_3^2)^{1/2}$ and interprets it relativistically. Finally he extends the results to systems of particles in generalized coordinates for the holonomic, scleronomic and stationary conservative case. *H. D. Block* (Ithaca, N. Y.).

Klein, Joseph. Liaisons d'Appell; espaces L et principe de moindre courbure. C. R. Acad. Sci. Paris 240, 2208–2210 (1955).

The author considers a conservative dynamical system, subjected to a constraint $f(x^i, t, x'^i)=0$, with $F_i=\mu\partial f/\partial x'^i$ as the corresponding force (liaison d'Appell). It is shown that the paths can be considered as geodesics in an L-space [see A. Lichnerowicz, Ann. Sci. Ecole Norm. Sup. (3) 62, 339–384 (1945), p. 350; MR 8, 352]. Above that it appears that the system verifies a principle of least curvature (curvature of the actual path with respect to the corresponding free path). *J. Haantjes* (Leiden).

Haimovici, M. La géométrie des systèmes mécaniques non holonomes. Acad. Repub. Pop. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Ser. I. 5, no. 3–4, 49–84 (1954). (Romanian. Russian and French summaries)

The author considers a non-holonomic mechanical system. The equations of constraint are supposed to be of the form $a_{\sigma i}\dot{x}^i=0$ the $a_{\sigma i}$ being independent of time. Using non-holonomic coordinates some invariants of the motion are

found by methods of tensor analysis. The geometrical meaning of these invariants in the configuration space is discussed.
J. Haantjes (Leiden).

Masotti, Arnaldo. Sui moti centrali di un punto vincolato. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18**(87), 515–529 (1954).

The author analyzes the motion of a particle constrained to move on a smooth curve under the action of a force that together with the reaction yields in every point of the path a resultant central force. Such necessarily plane motions are reduced to the ordinary central motions in case the constraint becomes zero. In this paper the author derives the characteristic equation (corresponding to Binet's equation) for both problems (the determination of the force from the given path and the centre of force, as well as the determination of path from the given force and its centre), and gives fully solved illustrative examples. *T. P. Anđelić.*

Litvin-Sedoi, M. Z. On a complex gyroscopic effect. Vestnik Moskov. Univ. **9**, no. 10, 19–24 (1954). (Russian)

The system (which may have been suggested by guided-missile instrumentation) consists of $n+1$ symmetric gyros G_i of axis Ox_i ($i=0, 1, \cdots, n$) and common fixed mass center O, G_i being constrained to revolve relative to G_{i-1} about Ox_i which is perpendicular to Ox_{i-1}. There are no external forces and no dissipation (G_i and G_{i-1} may be, respectively, the rotor and stator of an electric motor). After indicating how the expression for kinetic energy may be obtained by means of matrix multiplication, the paper turns to the case $n=2$, and covers more than one page with the underived equations of motion, stated in ill-advised notation and ridden with substantial misprints. Let θ, ψ, ϕ be the Euler angles for G_0, and ϕ_i the angular displacement of G_i relative to G_{i-1}. It can be verified, without computing the kinetic energy, that ϕ and ϕ_2 are cyclic, but ϕ_1 is not. For reasons unstated, the author starts a rather confused search for values of $\dot\phi_2$ which will make some of the angular accelerations $\ddot\theta$, $\ddot\psi$, $\ddot\phi$ vanish instantaneously if $\dot\psi$ and $\dot\phi$ do so. But, if the paper's corrected equations are correct, it is obvious that $\dot\psi$, $\dot\phi$, $\dot\phi_1$, $\dot\theta$, $\dot\phi_2$ can vanish permanently provided that $(A_2-B_2)\theta \cos \phi_1+A_2\dot\phi_2=0$, A_2 being the axial, and B_2 the equatorial moment of inertia of G_2. *A. W. Wundheiler.*

Girardin, Pierre. Sur un effet secondaire du jet d'un engin auto-propulsé. C. R. Acad. Sci. Paris **240**, 843–845 (1955).

The author considers a self-propelled engine in rotation ω about an axis orthogonal to the longitudinal axis ξ of the engine. The body of the engine together with the masses for the propulsion forms a system of revolution and it is assumed that the flow of matter is continuous and parallel to the axis of symmetry of the whole system. By simple reasoning he then shows that the effect of ejection on the absolute movement of this system is: 1) the force of propulsion in the direction of the longitudinal axis; 2) a moment which is algebraically addible to the aerodynamical damping moment of the system; and 3) one supplementary force more, which is normal to both ω and ξ. All results about this third effect are upper limits. *T. P. Anđelić* (Belgrade).

Tesson, Fernand. Cinématique des systèmes à nombre de particules variable. C. R. Acad. Sci. Paris **240**, 845–847 (1955).

The author examines the system $S(t)$ with a variable number of particles enclosed by a surface $\Sigma(t)$ whose position with respect to a system of reference R is at every instant determined. The case when Σ is fixed with respect to R (e.g. to metallic parts of an engine) is of interest for problems of auto-propulsion. Conclusions are deduced by applying the mechanical laws to a system $S^*(\tau, t)$ which consists of the same particles and which at the instant τ coincides with the system $S(t)$. He considers the following vectorial functions

$$\mathbf{A}(t)=\int_{\gamma(t)} \rho\mathbf{a}(P, t)d\omega \quad\text{and}\quad \mathbf{A}^*(\tau, t)=\int_{\gamma^*(\tau,t)} \rho\mathbf{a}(P, t)d\omega$$

where $\gamma(t)$ resp. $\gamma^*(\tau, t)$ are the volumes enclosed by $\Sigma(t)$ and $\Sigma^*(\tau, t)$, ρ is the specific mass of the volume element $d\omega$ and $\mathbf{a}(P, t)$ a vectorial function defined at every instant t at the point P. He derives then the relations between the expressions $\partial\mathbf{A}^*(\tau, t)/\partial t$, resp. $\partial^2\mathbf{A}^*(\tau, t)/\partial t^2$, for $t=\tau$, on one side and the velocity and acceleration of the vectorial function $\mathbf{A}(t)$ on the other side. The derived formulas are then used for conclusions on the centres of gravity $G(t)$ and $G^*(\tau, t)$ of both systems considered. He shows that in the case of a fixed Σ the obtained relations coincide with the ones obtained by P. Girardin in the paper reviewed above.
T. P. Anđelić (Belgrade).

Tesson, Fernand. Dynamique des systèmes à nombre de particules variable. Application à l'autopropulsion. C. R. Acad. Sci. Paris **240**, 1050–1052 (1955).

This paper is the direct continuation of the paper reviewed above. Here are analyzed the absolute motions of the centres of gravity G and G^* of the systems $S(t)$ and $S^*(\tau, t)$, as well as the motions of the systems about their centres of gravity. The author shows that, in the case of an enclosure fixed with respect to an engine, one obtains the result of P. Girardin in the paper reviewed second above. Finally, he points out an extension of the mentioned results of Girardin.
T. P. Anđelić (Belgrade).

Hydrodynamics, Aerodynamics, Acoustics

★**Birkhoff, Garrett. Hydrodynamics. A study in logic, fact, and similitude.** Dover Publications, Inc., New York, 1955. xiii+186 pp. Clothbound: $3.50; paperbound: $1.75.

Reprint by photo-offset of the original edition [Princeton, 1950; MR 12, 365].

Vâlcovici, V. Les lignes de courant et les lignes de tourbillon dans le mouvement permanent d'un fluide idéal, barotrope. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **5**, 147–154 (1953). (Romanian. Russian and French summaries)

The author rediscovers Lamb's generalization of Bernoulli's equation (1878), and, following without acknowledgment a suggestion of T. Craig (1881), derives some relations concerning flux and vorticity in the case when the Lamb surfaces may be taken as coordinate surfaces in a triply orthogonal system. [Cf. the reviewer, Kinematics of vorticity, Indiana Univ. Press, 1954, §72.] For this special case, the author constructs a pair of functions which play the role of potential function and stream function.
C. Truesdell.

Lüst, R., und Schlüter, A. Eine spezielle Art nichtwirbelfreier Lösungen der hydrodynamischen Gleichungen. Z. Angew. Math. Mech. **35**, 45–47 (1955). (English, French and Russian summaries)

The authors obtain some steady, rotational flows of an incompressible non-viscous fluid for which the vortex lines coincide with the streamlines. *J. B. Serrin.*

Ergun, A. N. Self-superposable fluid motions. Comm. Fac. Sci. Univ. Ankara. Sér. A. **6**, 89–151 (1954). (Turkish summary)

A fluid motion is self-superposable if its velocity field when doubled remains a solution of the equations of motion, with adjusted pressure [for conditions and generalizations, cf. §95 of the reviewer's The kinematics of vorticity, Indiana Univ. Press, 1954]. The author's objective is to obtain self-superposable motions of a viscous incompressible fluid subject to conservative extrinsic force. For such a fluid, a condition for self-superposability is curl $(v \times \operatorname{curl} v)=0$. The author constructs a goodly number of solutions in terms of familiar functions. Many of these solutions are already known, sometimes in more general form, but they are here derived from a unified standpoint. In all cases the pressure is exhibited. Solutions included are steady and unsteady, plane, axially-symmetric, pseudo-plane, and three-dimensional. *C. Truesdell* (Bloomington, Ind.).

Bloh, È. L. Influence of depth of immersion of a sphere on the coefficient of added mass in horizontal impact. Prikl. Mat. Meh. **19**, 353–358 (1955). (Russian)

In a previous paper [Prikl. Mat. Meh. **17**, 579–592 (1953); MR 16, 82] the author has studied the motion due to horizontal impact on a sphere half immersed in an ideal fluid which fills a lower half-space. The present paper considers, by a similar method, the case of a sphere totally immersed to a finite depth. The coefficient of added mass is computed as a function of depth of immersion. *R. Finn.*

McLeod, E. B., Jr. The explicit solution of a free boundary problem involving surface tension. J. Rational Mech. Anal. **4**, 557–567 (1955).

For two-dimensional, irrotational, steady flow of an incompressible fluid, the author finds an explicit expression for the motion of an isolated bubble influenced by surface tension. The method of solution is based on a conformal mapping which has no direct physical interpretation, in contrast with the hodograph transformation familiar from classical free-boundary theory. The shape of the free surface of the bubble is given by the formula $z=\zeta-2/3\zeta-1/27\zeta^3$, with $|\zeta|=1$. A variational formulation of the flow problem involving the virtual mass and the length of the boundary is presented, and some related flow patterns bounded in part by fixed arcs are described. *P. R. Garabedian.*

Dolapčiev, Bl. On the stability and oblique flow of two-parameter vortex streets. Dokl. Akad. Nauk SSSR (N.S.) **98**, 349–352 (1954). (Russian)

L'A. considère la configuration C, formée de deux files parallèles de tourbillons ponctuels; C est caractérisée par les paramètres classiques: $a=h/l$ et $\lambda=d/l$ $(d\neq 0, d\neq \frac{1}{2}l)$. Au premier ordre, la condition de stabilité de C s'écrit:

$$\operatorname{sh}(\pi a)=\sin(\pi\lambda). \tag{1}$$

L'A. montre que, moyennant (1), la stabilité de C subsiste pour les perturbations les plus générales du second ordre; globalement, la file ne peut subir qu'un déplacement de translation. L'A. insiste sur cette conclusion qui contredit les prévisions de Maue, reprises par A. Sommerfeld, mais que confirment, dans une certaine mesure, les essais effectués à la demande de l'A. *J. Kravtchenko* (Grenoble).

Dolapčiev, Bl. On approximate determination of the eddy resistance. Dokl. Akad. Nauk SSSR (N.S.) **98**, 743–746 (1954). (Russian)

Une formule classique de Kármán donne la résistance frontale, éprouvée par un contour circulaire placé dans un courant plan avec des tourbillons régulièrement distribués suivant deux files parallèles. La relation en cause, valable aux faibles vitesses, contient six paramètres dont deux ne sont susceptibles que d'une détermination expérimentale. A. Kosmodemyanskiĭ a émis des hypothèses complémentaires permettant le calcul théorique des coefficients en cause. L'A. approfondit la question et précise la nature des configurations pour lesquelles ces hypothèses sont valables. *J. Kravtchenko* (Grenoble).

Golubev, V. V. On the structure of the confused zone behind a poorly streamlined body. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 12, 19–37 (1955). (Russian)

The author's definition of the confused zone appears to be a dead-water region behind the body. In this paper attention is confined to the circular cylinder and the confused zone is assimilated to the average field of the flow induced by a semi-infinite Kármán vortex street in which the ratio of the breadth to the spacing of the vortices is taken to have the theoretical value $0.281\cdots$ necessary for stability. The reviewer is unable to understand the numerical results given for the velocity field, which appear to have been obtained by simple addition of the velocities of the perfect fluid flow past a circular cylinder alone in the fluid and the field induced at one end of the semi-infinite street whose breadth is equal to the diameter of the cylinder. As no account is taken of viscosity it is difficult to see what relevance the calculations have to an attempt to reconcile the observed values with them. *L. M. Milne-Thomson* (Greenwich).

Murai, H. Theorie über die Gitterströmung beliebig geformter Flügelprofile mit grossen Wölbungs- und Dicken-Verhältnissen. Z. Angew. Math. Mech. **35**, 48–54; erratum 400 (1955). (English, French and Russian summaries)

The cascade is transformed into a single closed curve by a logarithmic transformation, and thence by the Kármán-Trefftz transformation into a pseudo-circle. This is finally mapped on a circle by the well-known iterative method of Theodorsen. The results of calculation by this method for a certain turbine cascade are presented in comparison with the measured pressure distribution. This is a concise presentation of the method. The reviewer suspects that much labor and a number of practical difficulties, such as have been encountered by others using similar procedures [e.g., W. Traupel, Sulzer Tech. Rev. **1945**, no. 1, 25–42; MR 7, 423], lie behind this work. *W. R. Sears.*

Nickel, K. Über spezielle Systeme von Tragflügelgittern. II. Theorie der dünnen Profile. Ing.-Arch. **23**, 102–118 (1955).

[For part I see Ing.-Arch. **22**, 108–120 (1954); MR 16, 86.] To calculate the plane flow field of a cascade of two-dimensional airfoils in incompressible flow, the author considers them to be replaced by distributions of vortices around their contours. For thin airfoils, the contour is then approximated by the chord line. The result can be inverted,

i.e., the pressure distribution calculated for given geometry, only in the special cases where the angle between cascade and chord lines is 0° and 90° (unstaggered cascade). These cases are worked out in some detail, including integral formulas for total lift and pitching moment, analogous to the well-known Munk integrals of single-airfoil theory. Systems of parallel cascades are considered next, and analogous results are obtained. As an example, a case of two rows of flat airfoils of different chords is worked out, as well as the problem of designing an unstaggered cascade for given circulation distribution. An example is also given of the calculation of pressure distribution on uncambered profiles at zero incidence in an unstaggered cascade. It is pointed out that the familiar approximations of thin-airfoil theory necessarily lead to absurdities if they are applied to cascades whose blade spacing is not much greater than the profile thickness. *W. R. Sears* (Ithaca, N. Y.).

Golubev, V. V. On the theory of a wing of low aspect ratio. Prikl. Mat. Meh. **19**, 143–158 (1955). (Russian)

The author's posthumously published note recalls that with decreasing aspect ratio the center of pressure of a wing moves rearward and the stalling angle increases. He conjectures that the inflow around the wing tips causes a levelling of the velocity distribution on the upper surface that accounts for these phenomena. To estimate inflow for a rectangular wing he uses a system of horseshoe vortices with all trailing vortices coming off the wing tips parallel to the undisturbed flow. The distribution of vortex intensity across the wing is taken to be that of the flat plate of the same chord in plane flow at the same angle of attack. The maximym velocity v_{max} on the upper surface is approximated by the plane-flow value, while the velocity v_0 at the trailing edge is the plane-flow value increased by an amount required to compensate for the inflow around the wing tips. An empirical criterion for stalling of large aspect ratio wings, (*) $v_{max}/v_0=1.2$, is then applied. If Joukowski-airfoil rather than flat-plate values of v_{max} and v_0 are used, then (*) yields an estimate of stalling angle as a function of thickness as well as aspect ratio, which shows the appropriate behavior and order of magnitude at low aspect ratios. *J. H. Giese* (Havre de Grace, Md.).

Fraeys de Veubeke, B. Aérodynamique instationnaire des profils minces déformables. Bull. Service Tech. de l'Aéronautique, Bruxelles, no. **25**, 108 pp. (1953).

This is a detailed survey of non-stationary airfoil theory for two-dimensional, incompressible flow. The author's original contributions include extensive partial fraction approximations to Theodorsen's circulation function, the corresponding, exponential approximations to Wagner's indicial function, and the analysis of an exponentially (positively or negatively) damped airfoil oscillation started from rest.

The reviewer notes that this last contribution provides perhaps the most complete and physically satisfactory resolution of the difficulties connected with the analytic continuation of the Theodorsen function [A. I. van de Vooren, J. Aero. Sci. **19**, 209–211 (1952); E. V. Laitone, ibid. **19**, 211–213 (1952); W. P. Jones, ibid. **19**, 213 (1952); MR **14**, 103]. *J. W. Miles* (Los Angeles, Calif.).

Rumyancev, V. V. On stability under the conditions of S. A. Čaplygin of the screw motion of a rigid body in a fluid. Prikl. Mat. Meh. **19**, 229–230 (1955). (Russian)

The paper applies Kirchhoff's equations [cf. Lamb, Hydrodynamics, 6th ed., Cambridge, 1932, ch. vi], for which he gives credit to Čaplygin, to the case when the kinetic energy T of the body-fluid system is given by

$$2T=(b+c)R_1^2+(b-c)R_2^2+bR_3^2+P_1^2+P_2^2+2P_3^2,$$

where (R_1, R_2, R_3) and (P_1, P_2, P_3) are, respectively, the projections on a body-bound frame $Ox_1x_2x_3$, of the linear impulse R and the angular impulse P. The weight of the body and the displaced fluid are equal, and the unused assumption is made that the mass centers of both are at the same distance from Ox_2x_3. Then R^2 and the scalar product $R\cdot P$ are constant, and the energy integral exists. The special case is considered when R and P have the same constant direction in Ox_2x_3, and, therefore, are both constant in space. This uniform screw motion is shown to be stable by Lyapunov's second method. [In Čaplygin's case $R\cdot P=0$.] *A. W. Wundheiler* (Chicago, Ill.).

Harlamov, P. V. A case of integrability of the equations of motion of a heavy rigid body in a fluid. Prikl. Mat. Meh. **19**, 231–233 (1955). (Russian)

The problem of the preceding review is here considered for the case of "helicoidal symmetry" [Lamb, loc. cit., p. 174, case 7°], i.e.,

$$2T=a(R_1^2+R_2^2)+2b(R_1P_1+R_2P_2)+c(P_1^2+P_2^2)\\ +fR_3^2+2gR_3P_3+hP_3^2$$

(the notation is the same as in the preceding review, not the paper's). The linear impulse is constant in the inertial frame. The assumption is made that it is vertical, which makes the vertical component of the angular impulse a constant. The energy integral exists, and a fourth integral is found stating that the axial component of the angular impulse is constant. The four integrals yield a solution in terms of elliptic functions, similar to that of the Lagrange-Poisson case. The special case of axial uniform screw motion is proved to be stable by the same method as in the preceding review. *A. W. Wundheiler* (Chicago, Ill.).

Ostapenko, V. M., Fil'čakov, P. F., and Šamans'kiĭ, V. Ê. On modelling of plane flows with circulation. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 16–20 (1955). (Ukrainian. Russian summary)

DeWitt, T. W. A rheological equation of state which predicts non-Newtonian viscosity, normal stresses, and dynamic moduli. J. Appl. Phys. **26**, 889–894 (1955).

The author proposes a constitutive equation which is a special case of the one introduced first by S. Zaremba [Bull. Internat. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. **1903**, 594–614]. He discusses briefly the flow between two rotating concentric cylinders [for a detailed investigation applying to more general materials see W. Noll, J. Rational Mech. Anal. **4**, 3–81 (1955); MR **16**, 764] and the flow between two rotating parallel plates. *W. Noll.*

Kiselev, A. A. Solution of the linearized equations of unsteady flow of a viscous incompressible fluid in an unbounded region. Dokl. Akad. Nauk SSSR (N.S.) **101**, 43–46 (1955). (Russian)

Soient: Ω, un domaine borné, simplement connexe (de l'espace ordinaire x_1, x_2, x_3) dont S est la frontière; $Q=\Omega\times[0\leq t\leq l]$; $\mathbf{f}(x_i, t)$ un vecteur défini sur Q dont les composantes εL_2 sur Q (propriété qu'on notera dans la suite: $\mathbf{f}\varepsilon L_2(Q)$; H, l'espace des $\mathbf{f}$ sur lequel on a défini le produit scalaire $(\mathbf{f}, \varphi)=\int\mathbf{f}\cdot\varphi dQ$. On sait qu'on a la décomposition: $H=J\oplus G$, où J et G sont deux sous-espaces ortho-

gonaux complémentaires de H, tels que tout $\mathbf{v} \varepsilon J$ est solénoïdal et tangent à la surface latérale du cylindre Q à la frontière et tout $\mathbf{g} \varepsilon G$ est de la forme grad p, $p(x, t)$ étant une fonction scalaire à dérivées premières en x_i continues.

Alors l'A. démontre le résultat suivant. Pour tout $\mathbf{f} \varepsilon H$, il existe un système et un seul de solution $\mathbf{u}$, p du système:

$$\frac{\partial \mathbf{u}}{\partial t} - \bar{\Delta} u + \text{grad}\, p = \mathbf{f}, \quad \text{div}\, \mathbf{u} = 0,$$

avec les conditions aux limites: $\mathbf{u}|_S = 0$, $\mathbf{u}|_{t=0} = 0$, tel que $\partial \mathbf{u}/\partial t \,\varepsilon\, L_2(Q)$, $\partial \mathbf{u}/\partial x \,\varepsilon\, L_2(Q)$ presque partout sur $0 \leqq t \leqq l$. Les propriétés de régularité de $\mathbf{u}$ et $\bar{\Delta} u$ sont précisées.

Ce résultat remarquable, établi en utilisant les récentes méthodes de O. Ladyženskaya, entraîne l'unicité et l'existence de la solution du système linéarisé de Navier dans le cas des données initiales homogènes. On peut l'étendre au cas où $\mathbf{u}|_{t=0} \neq 0$ et aux domaines Ω non bornés.

J. Kravtchenko (Grenoble).

Allen, D. N. de G., and Southwell, R. V. Relaxation methods applied to determine the motion, in two dimensions, of a viscous fluid past a fixed cylinder. Quart. J. Mech. Appl. Math. **8**, 129–145 (1955).

In this paper, the relaxational methods are applied to determine the steady, two-dimensional, laminar flow over a cylinder immersed in a uniform stream of infinite extent. As examples, the case of a circular cylinder is described in detail. The flow-patterns and the constant-vorticity contours are calculated at Reynolds numbers of 10, 100 and 1000. The calculated drag coefficient compares closely with the experimental values in the range of Reynolds numbers considered.

Y. H. Kuo (Ithaca, N. Y.).

Tarapov, I. E. Solution of a problem of motion of a viscous gas between two moving parallel plates with heat loss. Prikl. Mat. Meh. **19**, 325–330 (1955). (Russian)

Viscous gas flows steadily between two planes, each at a constant temperature, and one in uniform motion parallel to the other which is fixed. The pressure gradient in the direction of motion is assumed to be zero. Non-dimensional equations of motion and energy are then obtained in the forms

$$\frac{d}{dy}\left(T^m \frac{dv}{dy}\right) = 0, \quad \frac{d}{dy}\left(T^m \frac{d\Theta}{dy}\right) = 0,$$

where y is the coordinate perpendicular to the planes and v, T, Θ are respectively velocity, temperature and total head temperature. This reduction depends on the viscosity being proportional to a power T^m, and on the (unstated) assumption that the specific heat c_p is independent of temperature. It follows at once that T is a quadratic function of v, and then y is found as an incomplete beta-function of v. The results are applied to several particular cases of the boundary conditions.

L. M. Milne-Thomson (Greenwich).

Kanwal, R. P. Rotatory and longitudinal oscillations of axi-symmetric bodies in a viscous fluid. Quart. J. Mech. Appl. Math. **8**, 146–163 (1955).

The author calculates approximately the motion arising from small rotatory or longitudinal oscillations of an axially symmetric body in an infinite mass of fluid which is at rest at infinity. By means of the Stokes stream function, the solutions to the linearized Navier-Stokes equations are obtained for the cases of the sphere, infinite circular cylinder, prolate and oblate spheroids and the circular disk.

Y. H. Kuo (Ithaca, N. Y.).

Barenblatt, G. I. On approximate solution of problems of one-dimensional unsteady filtration into a porous medium. Prikl. Mat. Meh. **18**, 351–370 (1954). (Russian)

The gas pressure in the author's problem satisfies Boussinesq's equation $\partial p/\partial t = a^2 \partial^2 p^2/\partial x^2$. Taking $p(x, 0) = 0$, $p(0, t) = P_1 = \text{const.}$, and $\lim_{x\to\infty} p(x, t) = P = \text{const.}$, this equation has the solution

$$p(x, t) = P_1 F(\xi, \lambda), \quad \xi = x/a(P_1 t)^{1/2}, \quad \lambda = P/P_1,$$

where $F(\xi, \lambda)$ satisfies $\partial^2 F/\partial \xi^2 + \frac{1}{2}\xi \partial F/\partial \xi = 0$, with conditions $F(0, \lambda) = 1$, $F(\infty, \lambda) = \lambda$. The author allows P_1 to vary, $P_1 = \Phi(t)$, and writes

$$p(x, t) \doteq \Phi(x, t) F\left[\frac{x}{a(\Phi(t)\sigma(t))^{1/2}}, \frac{P}{\Phi(t)}\right]$$

as an approximate solution. For arbitrary $\sigma(t)$ the initial and boundary conditions are satisfied, and if

$$\sigma(t) = [\Phi^2(t) - P^2]^{-1} \int_0^t [\Phi^2(t) - P^2] dt,$$

the integral relation

$$\frac{d}{dt}\int_0^\infty x[p(x, t) - P] dx = a^2 p^2(0, t) - a^2 P^2$$

is also satisfied. This latter equation is derived from the integral relation

$$\int_0^\infty x(\partial p/\partial t) dx = \int_0^\infty x(a^2 \partial^2 p^2/\partial x^2) dx.$$

This theme is carried out also for axially symmetric and spherically symmetric one-dimensional problems, and some special cases are given in greater detail as illustrations.

R. E. Gaskell (Seattle, Wash.).

Nužin, M. T. On the formulation and solution of inverse problems of forced filtration. Dokl. Akad. Nauk SSSR (N.S.) **96**, 709–711 (1954). (Russian)

A brief, general discussion of the inverse method for filtration problems, mentioning possible applications but giving none. Chapter 5 of P. Ya. Polubarinova-Kočina, Theory of motion of ground water [Gostehizdat, Moscow-Leningrad, 1952; MR **15**, 71] completely overwhelms this note, in the reviewer's opinion.

R. E. Gaskell.

Nužin, M. T. On the solution of some problems of filtration under pressure. Inžen. Sb. **18**, 49–60 (1954). (Russian)

Filtration under dams situated upon a porous medium over an impermeable layer is studied with the help of the Schwarz-Christoffel transformation. Several examples are given. Again, the author appears to have been anticipated by Polubarinova-Kočina [Theory of motion of ground water, Gostehizdat, Moscow-Leningrad, 1952, ch. 3, 6; MR **15**, 71].

R. E. Gaskell (Seattle, Wash.).

Glauert, M. B., and Lighthill, M. J. The axisymmetric boundary layer on a long thin cylinder. Proc. Roy. Soc. London. Ser. A. **230**, 188–203 (1955).

In this paper the laminar boundary layer in an axial flow about a long thin cylinder is investigated. A series solution of this problem which is valid near the nose of the cylinder has previously been given by Seban and Bond [J. Aero. Sci. **18**, 671–675 (1951)]. Numerical corrections to this solution were given by Kelly [ibid. **21**, 634 (1954)]. The present investigation obtains results for the skin friction, the dis-

placement area, and the momentum-defect area valid along the length of the cylinder. First, a Pohlhausen method, based on a velocity profile which is quite accurate near the surface, is used. Secondly, far downstream when the boundary-layer thickness has become large compared to the cylinder radius an asymptotic series solution is found. The Pohlhausen solution shows good agreement with the Seban, Bond and Kelly solution near the nose and with the asymptotic solution far from the nose. Recommended values of the above mentioned parameters are given in graphical and tabular form as a function of $\nu x/Ua^2$ over the range $-3<\log_{10}(\nu x/Ua^2)<3$. Here a is the radius of the cylinder and U is the free stream velocity. The results are expected to be correct to within 2%. *R. C. DiPrima.*

Černyĭ, G. G. Laminar motion of gas and liquid in a boundary layer with a surface of discontinuity. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 12, 38–67 (1955). (Russian)

In many cases where gas or vapour flows along a solid surface there is formed a layer with a well-defined boundary across which there is a jump in the state of the medium as when, for example, a jet of steam is separated by a film of water from a cooling surface over which the steam flows, or an injection of liquid is employed to protect a wall from the flow of a hot gas. Such problems can be treated by the mathematical methods of boundary-layer theory. The author sets up the equations of laminar flow taking into account viscosity and heat conduction and applies them to (1) streaming over a semi-infinite porous plate, (2) steam moving over a plane on which it condenses.

L. M. Milne-Thomson (Greenwich).

Millsaps, Knox. The Obukhoff spectrum of homogeneous isotropic turbulence. J. Aero. Sci. **22**, 511 (1955).

Betti, Ezio. A new method of solution of equation of compressible flow. J. Aero. Sci. **22**, 516 (1955).

Bergman, Stefan. Tables for the determination of fundamental solutions of equations in the theory of compressible fluids. Math. Tables Aids Comput. **9**, 8–14 (1955).

The author [Trans. Amer. Math. Soc. **62**, 452–498 (1947); MR **10**, 162] has shown how to obtain fundamental solutions of the linear differential equations of subsonic flow for the stream-function and velocity-potential in the pseudo-logarithmic plane. Here, tables are given of the real and imaginary parts of functions required for the computation of these fundamental solutions. *D. C. Pack.*

Grib, A. A. On a particular solution of the equations of plane, cylindrical, and spherical waves. Dokl. Akad. Nauk SSSR (N.S.) **102**, 225–227 (1955). (Russian)

A particular solution of the equations of motion, continuity and entropy for non-steady motion of an ideal gas in a tube is discussed. Let u, a, P, and ρ denote the velocity, velocity of sound, pressure, and density respectively. The solution found is of the separable form $u=T(t)V(r)$, $a=T(t)C(r)$, $\theta=T_1(t)S(r)$ in which $\theta^k=P/\rho^k$ and $k=c_p/c_v$, while $T(t)=(1+mt)$, $T_1(t)=(1+mt)^{-q}$. In the cylindrical and spherical cases the motion can be constructed from a non-stationary source whose intensity is determined. In the planar case, with $q=0$, $V(r)$ is found as a function of $C(r)$.

N. D. Kazarinoff (Lafayette, Ind.).

Aslanov, S. K. On the amount of the local supersonic zone in the flow of a compressible gas about a wedge. Prikl. Mat. Meh. **19**, 359–362 (1955). (Russian)

The author continues the work of Cole [J. Math. Phys. **30**, 79–92 (1951); MR **15**, 263] on flow of a slightly subsonic stream past a wedge consisting of a nose section of small apex angle and a rear section with faces parallel to the undisturbed flow direction. A local supersonic zone of flow is centered on the shoulder of the wedge and its nature is calculated. As in Cole's work, the sonic line is assumed to extend vertically upwards from the shoulder. The vertical velocity component is plotted against distance along the sonic line for a number of values of the transonic similarity parameter, K_1. The maximum height of the supersonic zone is also plotted against this parameter.

Since Cole's paper was written Trilling and Walker [ibid. **32**, 72–79 (1953); MR **15**, 263] have developed a solution of the basic problem without assuming that the sonic line is vertical. It would be useful if the present calculations were extended to take account of this more realistic treatment.

M. Holt (Cambridge, Mass.).

Yur'ev, I. M. On the linearized theory of the supersonic flow of a gas about bodies of revolution. Prikl. Mat. Meh. **19**, 363–367 (1955). (Russian)

By Laplace's cascade method one can find the general solution of

$$(*)\qquad \partial^2 z/\partial\xi\partial\eta+a(\partial z/\partial\xi+\partial z/\partial\eta)=0,$$

where $a=B/\sinh[B(\xi+\eta+C)]$, and B and C are constants. The author prescribes a choice of B and C for which z can be used to approximate the perturbation velocity potential ϕ_1 for axisymmetric supersonic flow, which satisfies

$$\partial^2\phi_1/\partial\xi\partial\eta+0.5(\xi+\eta)^{-1}(\partial\phi_1/\partial\xi+\partial\phi_1/\partial\eta)=0$$

for characteristic variables: $2\xi=-x+\beta r$, $2\eta=x+\beta r$, and $\beta^2=M_\infty^2-1$. By integrating a first-order linear ordinary differential equation one can determine the two arbitrary functions in the general solution z of (*) to approximate the solution of the following problem: (i) $\phi_1=\phi_*(\eta)$ on a characteristic $\xi=\xi_0$; (ii) $\partial\phi_1/\partial r=w_\infty dr/dx$ on a boundary $r=r(x)$, where w_∞ is the undisturbed speed of flow. For a parabolic boattail pressure distributions obtained by use of the author's method and by the method of characteristics agree very closely except toward the end of the boattail.

J. H. Giese (Havre de Grace, Md.).

Kuo, Y. H. A similarity rule for the interaction between a conical field and a plane shock. J. Aero. Sci. **22**, 504–505 (1955).

Kawamura, Ryuma, and Oguchi, Hakuro. Curved shocks in pseudostationary flows. J. Aero. Sci. **22**, 210–211 (1955).

A. H. Taub [Ann. of Math. (2) **58**, 501–527 (1953); MR **15**, 839] has derived expressions for the ratio of the curvature of a curved shock in a pseudo-stationary flow to the curvature at the shock of a pseudo-streamline and for the ratio of the pressure gradients immediately behind the shock to the curvature of the shock. In this note the authors have applied these relations to several particular cases showing, among other similar results, that the Mach and reflected shocks have zero curvature at the triple point of a Mach reflection.

P. Chiarulli (Washington, D. C.).

Oguchi, Hakuro. On the attached curved shock in front of an open-nosed axially symmetrical body. J. Phys. Soc. Japan **9**, 861–866 (1954).

The author considers the flow behind a shock attached to the outer part of the nose of an open-nosed axially symmetrical body in the immediate neighborhood of the nose. The equations of motion are linearized by means of a perturbation of the state immediately behind the shock at the nose of the body and solutions are found for the first-order terms of a power-series expansion in the radial distance from the nose. In addition, an expression is found for the initial curvature of the shock in terms of the principal curvatures at the nose and the initial inclinations of the shock and body. *P. Chiarulli* (Washington, D. C.).

Karlikov, V. P. Solution of the linearized axially symmetric problem of a point explosion in a medium with variable density. Dokl. Akad. Nauk SSSR (N.S.) **101**, 1009–1012 (1955). (Russian)

An intense explosion is released in an atmosphere in which the density varies slowly with height, according to the relation,

$$\rho_1=\rho_{01}-\epsilon_z^{\kappa}=\rho_{01}-\epsilon r^{\kappa}\cos^{\kappa}\theta,$$

where ρ_{01}, ϵ and κ are constants (ϵ small), Oz is the vertical axis, r is the radial distance from the centre of the explosion, and θ is the latitude measured from the vertical. The disturbance due to the explosion is represented as the sum of two parts, a symmetrical part, the properties of which are known from previous work on explosions in a uniform region, and a small unsymmetrical contribution representing the effects of variation in density. The additional disturbance is found from the solution of a set of linear partial differential equations with independent variables $\lambda=E_0\rho_{01}^{-1}t^2r^{-5}$, and θ, where E_0 is the energy released by the explosion and t is the time from initiation.

If $V_r(\lambda,\theta)$, $V_\theta(\lambda,\theta)$, $R(\lambda,\theta)$, $P(\lambda,\theta)$ are non-dimensional variables defining the additional velocity components, density and pressure respectively, solutions of these equations in the form $V_r(\lambda,\theta)=\mathcal{R}(\theta)V_r(\lambda)$, $V_\theta(\lambda,\theta)=V_\theta(\theta)V_\theta(\lambda)$, $R(\lambda,\theta)=\mathcal{R}(\theta)R(\lambda)$, $P(\lambda,\theta)=\mathcal{R}(\theta)P(\lambda)$ are sought. It is found that $\mathcal{R}(\theta)$ and $V_\theta(\theta)$ are Legendre functions and the functions $V_r(\lambda)$, $\cdots$ satisfy a set of ordinary equations of non-standard type. The functions are chosen to satisfy the boundary conditions at the shock wave (simplified on the assumption that the shock strength is very large) and the condition that the additional disturbance vanishes at the centre of the explosion. The symmetrical solution is based on a similarity hypothesis and is stated to have been developed by Sedov [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **52**, 17–20 (1946)]. It is of interest that a solution of the same type was developed independently by Taylor [Proc. Roy. Soc. London. Ser. A. **201**, 159–174 (1950)]. *M. Holt* (Cambridge, Mass.).

Asaka, Saburô. Application of the thin-wing-expansion method to the flow of a compressible fluid past a symmetrical circular arc aerofoil. J. Phys. Soc. Japan **10**, 482–492 (1955).

This is a painstaking analysis of subsonic compressible flows past circular-arc aerofoils at zero incidence. It is based on Imai's procedure, in which the successive terms of thickness-parameter expansions of velocity potential and stream function are determined by a complex-variable method. The third approximations are given (in a series in which the Prandtl-Glauert law corresponds to the first approximation). For a thickness-chord ratio of 10 percent, the method yields satisfactory results up to $M=0.7$ while convergence is less good for $M=0.8$. The discussion contains an interesting assertion, credited in part to Fukatsu, according to which the velocity distribution over the upper surface of an aerofoil in subsonic flow is almost independent of the shape of the lower surface and vice versa. *A. Robinson* (Toronto, Ont.).

Baños, Alfredo, Jr. Fundamental wave functions in an unbounded magneto-hydrodynamic field. I. General theory. Phys. Rev. (2) **97**, 1435–1443 (1955).

The propagation of waves in an infinite inviscid compressible medium in the presence of a uniform magnetic field is considered. In the treatment the terms in conductivity and in displacement current (in Maxwell's equations) are retained. In this way the author is able to obtain pure electromagnetic waves, Alfven waves and sound waves as appropriate limiting cases. Explicit solutions for all the field quantities are obtained for plane as well as for cylindrical waves. *S. Chandrasekhar* (Williams Bay, Wis.).

Nardini, Renato. Su particolari campi alternativi della magneto-idrodinamica. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **89**, 17–36 (1955).

This paper deals with the propagation of sinusoidal waves in an ideal homogeneous incompressible fluid which fills the half-space $z>0$ in rectangular cartesian coordinates and which has constant density ρ, dielectric constant ϵ, conductivity γ, and permeability μ. It deals only with the case when the electric vector $\mathbf{E}$, the magnetic vector $\mathbf{H}$, the velocity vector $\mathbf{v}$ and the pressure p depend only on z and on the time t.

The phenomenon is governed by the equations

$$\operatorname{curl}\mathbf{H}=\gamma(\mathbf{E}+\mu\mathbf{v}\times\mathbf{H})+\epsilon\dot{\mathbf{E}},\quad \operatorname{curl}\mathbf{E}=-\mu\dot{\mathbf{H}},\quad \operatorname{div}\mathbf{H}=0,$$
$$\rho\dot{\mathbf{v}}=-\operatorname{grad}p+\mu\gamma(\mathbf{E}+\mu\mathbf{v}\times\mathbf{H})\times\mathbf{H},\quad \operatorname{div}\mathbf{v}=0.$$

It readily follows that $v_z=0$, that H_z is a constant H_0 ($\neq 0$, say) and that $v_xH_y=v_yH_x$. The equations to be solved split up into two groups, both of the form

$$-\frac{\partial H}{\partial z}=\gamma E+\epsilon\frac{\partial E}{\partial t}+\mu\gamma H_0 v,$$
$$\frac{\partial E}{\partial z}=-\mu\frac{\partial H}{\partial t},$$
$$\rho\frac{\partial v}{\partial t}=-\mu\gamma H_0E-\mu^2\gamma H_0^2v,$$

where $H=H_y$, $E=E_x$, $v=v_y$, in the first group, and $H=H_x$, $E=-E_y$, $v=v_x$, in the second group. These equations are solved in the usual way by taking $E=\mathcal{E}(z)e^{i\omega t}$, $H=\mathcal{H}(z)e^{i\omega t}$ and finally taking real parts. The solutions of the two groups are not independent; the functions H_x and H_y must either be in phase or have a phase difference π. *E. T. Copson.*

Lehnert, B. Magnetohydrodynamic waves under the action of the Coriolis force. II. Astrophys. J. **121**, 481–490 (1955).

It is known that in an incompressible inviscid fluid of infinite electrical conductivity, transverse plane waves of arbitrary amplitude can be propagated with the Alfvén velocity $V=H/(4\pi\rho)^{1/2}$. The author shows that his earlier results [Astrophys. J. **119**, 647–654 (1954); MR **15**, 1008] on the propagation of hydromagnetic waves of infinitesimal amplitude in a rotating fluid are valid as they stand even

for finite amplitudes. However, in contrast to the non-rotating case, an arbitrary finite disturbance cannot be expressed as a Fourier integral over the plane-wave solutions: the non-linear terms of the equations of motion prevents this. The anisotropic group velocity of the waves is also discussed. *S. Chandrasekhar* (Williams Bay, Wis.).

Lüst, R. Stationäre magnetohydrodynamische Stosswellen beliebiger Stärke. Z. Naturf. **10a**, 125–135 (1955).

The equations of hydromagnetic shocks derived in an earlier paper [Z. Naturf. **8a**, 277–284 (1953); MR **15**, 271] are used to obtain explicity the dependence on the Mach number of the jumps in pressure and density as well as the change in the direction of the magnetic field which occur at the shock front. It is shown that there are three distinct modes of propagation which in the limit of zero amplitude tend to the three known modes (the modified sound, the modified Alfven and the Alfven) of hydromagnetic wave propagation in a compressible medium [cf. H. C. van de Hulst, Problems of Cosmical Aerodynamics, Central Air Documents Office, Dayton, Ohio, 1951, chap. 6, p. 47; MR **13**, 399]. If c_-, c_A and c_+ denote the velocities of shock propagation for the three modes, then the mode corresponding to c_+ exists for every Mach number $M_+ \geqq 1$. On the other hand, the other two modes exist only for the ranges $1 < M_- \leqq a$, and $c_A/c_- < M_- \leqq a$, respectively. For sufficiently large Mach numbers these latter modes do not occur; also the domain of occurrence of these modes becomes smaller, the weaker the magnetic field. The results of the numerical computations are illustrated by a number of graphs. [Similar results have also been obtained by H. L. Helfer, Astrophys. J. **117**, 177–199 (1953); MR **14**, 804.]

S. Chandrasekhar (Williams Bay, Wis.).

Michael, D. H. The stability of a combined current and vortex sheet in a perfectly conducting fluid. Proc. Cambridge Philos. Soc. **51**, 528–532 (1955).

A plane surface of discontinuity of longitudinal velocity and magnetic field is assumed to exist in a perfectly conducting incompressible fluid of uniform density. The stability of this configuration to small two-dimensional disturbances at the surface of discontinuity is investigated and it is shown that the criterion for stability is

$$\frac{\mu}{2\pi\rho}(H_0^2+H_1^2) > (U_0-U_1)^2,$$

where H_0 and H_1 are the intensities of the magnetic fields on the two sides where the velocities are U_0 and U_1, respectively, and μ is the magnetic permeability and ρ the density.

S. Chandrasekhar (Williams Bay, Wis.).

Chandrasekhar, S. The instability of a layer of fluid heated below and subject to the simultaneous action of a magnetic field and rotation. Proc. Roy. Soc. London. Ser. A. **225**, 173–184 (1954).

This paper is devoted to an examination of the stability of a horizontal layer of fluid heated from below, and subject to an effective gravity g acting in the direction of the vertical, a magnetic field H and the Coriolis force resulting from a rotation Ω. When the vectors g, H, and Ω are parallel, the critical Rayleigh number R_c at which convection sets in depends on H and Ω through the nondimensional parameters $Q=\mu^2H^2d^2\sigma/\rho\nu$ and $T=4\Omega^2d^4/\nu^2$, where μ, σ, and ν are the coefficients of magnetic permeability, electrical conductivity, and kinematic viscosity, respectively, and d is the depth of the layer under consideration. In the case of a liquid confined between two free boundaries, the dependence of R_c on Q and T is explicitly established.

The results reveal some rather unexpected features. Thus if we start with an initial situation in which $T=10^6$ and no magnetic field is present, and then gradually increase the strength of the magnetic field, the turbulent cells which appear first at marginal stability will be elongated; but when the magnetic field has increased to a value corresponding to $Q=100$, cells of two very different sizes will appear simultaneously: one being elongated very much more than the other. As the magnetic field increases still further, the critical Rayleigh number will start decreasing and pass through a minimum; and eventually the inhibition due to magnetic field will predominate. Furthermore, for $T>500$, the critical Rayleigh number always shows an initial decrease with Q; only for $T>2500$ this decrease becomes very pronounced when Q passes through the critical value when two sets of cells appear simultaneously. For $T<400$, R_c appears to be a monotonically increasing function of Q.

Z. Kopal (Manchester).

Kaplan, S. A. Spectral theory of gas-magnetic isotropic turbulence. Ž. Eksper. Teoret. Fiz. **27**, 699–707 (1954). (Russian)

The aim of the paper under review has been to present certain solutions of the equations governing the energy distribution in the spectrum of an (isotropic) hydromagnetic turbulence; the equations themselves having been previously discussed by the author [Dokl. Akad. Nauk SSSR (N.S.) **94**, 33–36 (1954); MR **15**, 1001]. In the case of a stationary solution, the spectrum of the kinetic as well as magnetic energy is continuous, and its slope is characterized by an exponent α in the law $F(k)\sim k^{-\alpha}$ ($k=2\pi/l$, l being the size of a turbulent element) in the neighbourhood of 1.6, depending but slightly on the scale of the flow, velocity of sound, dissipation of energy through shock waves, etc.). If $\lambda>\nu$ ($\lambda=c^2/4\pi\sigma$, σ being the coefficient of electrical conductivity, and ν the viscosity coefficient), a second solution may apply, in which the exponent α in the spectrum of the kinetic energy is in the neighbourhood of 5/3, and in the spectrum of the magnetic energy, $\alpha\sim-1/3$. In the last part of the paper, the author applies his spectral theory to a study of nonstationary turbulence. *Z. Kopal.*

Skumanich, Andrew. On thermal convection in a polytropic atmosphere. Astrophys. J. **121**, 408–417 (1955).

A plane parallel polytropic atmosphere in which the pressure P is proportional to ρ^Γ (where ρ is the density) in equilibrium under a constant acceleration of gravity is considered. If the ratio of the specific heats of the gas is γ, then thermal stability of the atmosphere requires that $\gamma>\Gamma$. If $\gamma<\Gamma$, then the atmosphere is unstable and a measure of the degree of unstability is given by $\epsilon=(1-\gamma/\Gamma)/(\gamma-1)$. For $\epsilon>0$ the amplitude of any disturbance in the horizontal plane will increase exponentially with time. In this paper the characteristic-value problem for determining the e-folding time for a disturbance characterized by a wave number k is formulated and solved. It is shown that the instability increases indefinitely as $k\to\infty$; this is in sharp contrast to a homogeneous atmosphere [discussed by Rayleigh, Scientific papers, v. 6, Cambridge, 1920, p. 432] for which the e-folding time tends to a limiting value as $k\to\infty$. Solutions obtained by numerical integrations for the case $\epsilon=0.125$ are tabulated and illustrated. *S. Chandrasekhar* (Williams Bay, Wis.).

Heaps, H. S. Nonspecular-reflection of sound from a sinusoidal surface. J. Acoust. Soc. Amer. 27, 698–705 (1955).

The reviewer [same J. 26, 191–199 (1954); MR 15, 840] solved the problem of nonspecular reflection from a rough surface in the approximation that the deviation of the surface from a plane was everywhere small. The present paper attempts to extend this analysis by including higher-order terms in the amplitude of a sinusoidal surface. Unfortunately, the boundary condition applied by the author is zero pressure, rather than zero normal velocity, at the reflecting surface, so that his results are of only limited, physical interest.

J. W. Miles (Los Angeles, Calif.).

Jones, D. S. A critique of the variational method in scattering problems. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-78**, i+18 pp. (1955).

The problem of diffraction of sound waves by a rigid plane screen may be stated as follows: given a function f and an operator L, find a function g such that $f=Lg$. For when monochromatic sound waves are diffracted at an aperture S in a rigid plane screen, the velocity potential behind the screen is of the form

$$F(\mathbf{r}')=\frac{1}{4\pi}\int_S g(\mathbf{r})\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

and if $\mathbf{r}'$ is a point of the aperture, it is known that $F(\mathbf{r}')=f(\mathbf{r}')$, where $f(\mathbf{r}')$ is the velocity potential of the incident waves.

Often it is the amplitude of the far field which is of importance and not g itself. The amplitude A is the inner product (g, f') of g and some suitable incident field. For instance, the amplitude of the far field in the direction of the unit vector $\mathbf{n}$ is $\int_S g(\mathbf{r})e^{ik\mathbf{r}\cdot\mathbf{n}}d\mathbf{r}$. Thus in this and other scattering problems, we have to find the inner product $A=(g, f')$ when g is the solution of $Lg=f$, for a given linear operator L. If g' is such that $Lg'=f'$, then

$$A=(f', g)=(f, g')=(g', Lg).$$

This is the reciprocity theorem which exists for most scattering problems. Hence,

$$(*)\qquad A=\frac{(f', g)(f, g')}{(g, Lg')}.$$

The necessary and sufficient condition that $Lg=f$, $Lg'=f'$ is that the expression (*) should be stationary for independent small variation of g and g' about their correct values.

One method of solving this variational problem for the case $f'=f$, given by Levine and Schwinger, is to expand the fields in a set of functions and solve the simultaneous linear equations given by the variational principle. It is shown here that this is exactly equivalent to Galerkin's method of solving the integral equation $Lg=f$. This is extended to the more general expression (*) with $f'\neq f$. Another method of finding an approximate solution is to choose an expression for the field which is physically plausible and/or mathematically simple and insert it immediately in the variational expression. This is shown to be equivalent to choosing an approximation which satisfies the reciprocity theorem. It appears therefore that fundamentally the use of the variational method is to ensure that the reciprocity theorem is complied with.

E. T. Copson (St. Andrews).

Elasticity, Plasticity

Adkins, J. E. A note on the finite plane-strain equations for isotropic incompressible materials. Proc. Cambridge Philos. Soc. 51, 363–367 (1955).

Continuing earlier work by Adkins, Green, and Shield [Philos. Trans. Roy. Soc. London. Ser. A. 246, 181–213 (1953); MR 15, 369], the author obtains a relatively simple mathematical formulation for the general problem of finite plane strain of a Mooney material subject to no extrinsic force. He presents both a complex form and the following real form in rectangular co-ordinates:

$$\frac{\partial(x_1, \nabla^2x_1)}{\partial(y_1, y_2)}+\frac{\partial(x_2, \nabla^2x_2)}{\partial(y_1, y_2)}=0,\quad \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}=\lambda,$$

where the deformation is $y\rightarrow x$ and where λ is the normal stretch. The equations remain valid if x and y are interchanged, while λ is replaced by $1/\lambda$. These equations permit direct determination of the displacement. When the displacement is known, the stresses may be calculated in several ways; the author prefers use of Airy's stress function. As an example, he considers the following generalization of simple extension, simple shear, and pure shear:

$$x_1=x_1(y_1),\quad x_2=\lambda y_2/(dx_1/dy_1)+f(y_1),$$

for which he obtains the general solution. *C. Truesdell.*

Ericksen, J. L. Deformations possible in every compressible, isotropic, perfectly elastic material. J. Math. Phys. 34, 126–128 (1955).

In a previous paper [Z. Angew. Math. Phys. 5, 466–489 (1954); MR 16, 643] the author has attacked the problem of determining all deformations which are possible for any functional form of the strain-energy function in incompressible, isotropic, elastic materials when extraneous forces are absent. In this note the same problem for compressible bodies is solved. The simple solution is that the deformations in question must be homogeneous. *W. Noll.*

Nowiński, Jerzy, and Olszak, Wacław. On the principles of the theory of physically non-linear elastic bodies. Arch. Mech. Stos. 6, 139–168 (1954). (Polish. Russian and English summaries)

The authors propose a non-linear elastic stress-strain relation containing an additional second-order term in strain with an additional modulus, and develop on this basis one-dimensional equations of vibration and of bending, and some equations of plane strain. Considerations are limited to small strains, and do not refer to tensorial character of proposed relation, thus completely missing the "cross-elasticity" effects characteristic of the proposed relation.

A. M. Freudenthal (New York, N. Y.).

Mossakowska, Zofia. Stress functions for elastic bodies with three-axial orthotropy. Arch. Mech. Stos. 7, 87–96 (1955). (Polish. Russian and English summaries)

A set of three partial differential equations of sixth order is derived for the three stress-functions of an elastic body of three-dimensional orthotropy. In the case of isotropy these equations reduce to that of the Galerkin stress functions.

A. M. Freudenthal (New York, N. Y.).

Zapałowicz, Wiesław. Representation of a three-dimensional state of stress by means of the triangle of stress. Arch. Mech. Stos. 5 (1953), 641–652 (1954). (Polish. Russian and English summaries)

A three-dimensional state of stress defined by its principal stress-components σ_i is represented by a triangle in the two-

dimensional coordinate system (σ^2, σ); a parabola is fitted through these points. Utilizing the properties of the parabola, it is shown that this representation leads to a simple geometrical interpretation of the components (σ, τ) of the tractions on any arbitrary plane which is represented by the point $(\sigma^2+\tau_B^2, \sigma)$ inside the triangle.
A. M. Freudenthal (New York, N. Y.).

Fleĭšman, N. P., and Gnatikiv, V. M. Concentration of stress about a spherical cavity in a heavy elastic half-space. Dopovidi Akad. Nauk Ukrain. RSR 1954, 361–364 (1954). (Ukrainian. Russian summary)

Kupradze, V. D., and Bašeleĭšvili, M. O. New integral equations of the theory of elasticity of anisotropic bodies. Soobšč. Akad. Nauk Gruzin. SSR 15, 327–334 (1954). (Russian)

Kupradze, V. D., and Bašeleĭšvili, M. O. New integral equations of the anisotropic theory of elasticity and their application to the solution of boundary problems. Soobšč. Akad. Nauk Gruzin. SSR 15, 415–422 (1954). (Russian)

The authors investigate the first and second static boundary-value problems in anisotropic elasticity for plane simply connected domains by methods of potential theory. They show that the solutions can be represented by means of four types of potentials due to simple and double distributions. The unknown densities of distribution on the boundary can be determined either by solving the familiar singular integral equations of Cauchy's type or from Fredholm's equations with bounded kernels. The results pertaining to the existence and uniqueness of solution follow directly from the Fredholm theory. *I. S. Sokolnikoff.*

Wasiutyński, Z. Sur l'hypothèse de Jacques Bernoulli. Arch. Mech. Stos. 4 (1952), 93–103 (1953). (Polish. French summary)

It is attempted to show by a simple variational method that the equilibrium distribution of stress in the one-dimensional elastic problem of pure bending is governed by the condition of minimum potential energy both for linear and arbitrary non-linear elastic stress-strain relations.
A. M. Freudenthal (New York, N. Y.).

Zanaboni, Osvaldo. Sulla deformazione indotta dallo sforzo tagliante. Atti Accad. Sci. Ist. Bologna. Cl. Sci. Fis. Rend. (11) 1, no. 1, 13–22 (1954).

Engineering treatment is given of transverse shear in the problem of flexure with shear for prismatic bars of uniform cross-section. *H. G. Hopkins* (Sevenoaks).

Minasyan, R. S. Stretching of a composite prismatic bar with a slightly bent axis. Soobšč. Akad. Nauk Gruzin. SSR 15, 207–214 (1954). (Russian)

The author applies the theory developed by P. M. Riz [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 24, 110–113, 229–232 (1939); MR 2, 176] and A. K. Rukhadze [Soobšč. Akad. Nauk Gruzin. SSR 14, 525–532 (1953); MR 2, 176] to study the effect of the curvature of a slightly bent composite prismatic rod of fixed cross section on the state of stress, when the rod is in tension. *I. S. Sokolnikoff.*

Rüdiger, D. Dehnungsspannungen und Verschiebungen der Konoidschalen. Österreich. Ing.-Arch. 9, 37–44 (1955).

Der Verf. betrachtet Konoidschalen mit rechteckigen Grundrissen unter Randbelastungen und unter einer besonderen Art der Flächenbelastung. Zur Bestimmung von Dehnungsspannungen und Verschiebungen solcher Schalen verwendet er Neuber's Methode [Z. Angew. Math. Mech. 29, 97–108, 142–146 (1949); MR 12, 219].
T. P. Anđelić (Belgrad).

Kennard, E. H. Cylindrical shells: energy, equilibrium, addenda, and erratum. J. Appl. Mech. 22, 111–116 (1955).

The author reconsiders his former work [same J. 20, 33–40 (1953); MR 14, 817] and now believes more terms should be retained. He calculates the mean strain energy from the linear theory of elasticity by expanding in powers of the shell thickness and retaining or discarding terms according to a fixed scheme. He discusses conditions under which further terms can be neglected. There is no mention that the literature abounds in similar work. *C. Truesdell.*

Alumyaè, N. A. On the theory of axially symmetric deformation of shells of rotation with finite displacements. Prikl. Mat. Meh. 16, 419–428 (1952). (Russian)

After deriving equations of equilibrium for axisymmetric deformation of shells of revolution, the author obtains a corresponding variational equation which is intended to be useful in obtaining approximate solutions to problems which are difficult to treat by other known methods.
J. L. Ericksen (Washington, D. C.).

Koiter, W. T. On the diffusion of load from a stiffener into a sheet. Quart. J. Mech. Appl. Math. 8, 164–178 (1955).

The physical problem, one that is fundamental in aircraft stress analysis, is that of determining the shearing stress between a long stiffener and an elastic sheet, in terms of a constant longitudinal force P_0 applied at one end of the stiffener. For a stiffener of semi-infinite length $(x\geqq 0)$ attached to either an infinite sheet or to the edge of a semi-infinite sheet, the required shear stress $\tau_0(x)$ satisfies the equation

$$-2\pi^{-1}\int_0^\infty (\xi-x)^{-1}\tau_0(\xi)d\xi+\int_0^x \tau_0(\xi)d\xi=1$$

and the condition $\int_0^\infty \tau_0(\xi)d\xi=1$, all written in terms of non-dimensional variables. The Cauchy principal value, at $\xi=x$, is to be used in evaluating the first of the above integrals. The Mellin integral transformation is used to transform the above problem in integral equations into a problem in difference equations in the transform $T_0(s)$ of $\tau_0(x)$, where s is the complex parameter in the transformation. The difference equation is reduced to one with constant coefficients with the aid of the logarithmic derivative, and the resulting equation is solved by means of the Laplace transformation. Residue theory is used in making the required inverse transformations. The final form of the solution, from which a graph is drawn, is a series representation of the axial load $P(x)$ in the stiffener, where $P(x)$ and $\tau_0(x)$ are related by the equation $P(x)/P_0=1-\int_0^x \tau_0(\xi)d\xi$. Careful attention is paid to all details of the analysis. *R. V. Churchill.*

Levi, Beppo. Essay on the calculation of the deflection of thin plates. II. Math. Notae 14, 1–31 (1954). (Spanish)

[For part I see Math. Notae 12–13, 79–193 (1954); MR 16, 646.] By means of biharmonic polynomials, subjected to pointwise conditions on the boundary of a rec-

tangle, empirical approximations are indicated for the boundary-value problems of rectangular thin plates.
G. Fichera (Trieste).

Tiffen, R. Some problems of thin clamped elastic plates. Quart. J. Mech. Appl. Math. **8**, 237–250 (1955).

Analysis, based upon theory of functions of complex variables, is given of certain problems of the small transverse displacement of a thin, uniform, clamped, elastic plate in equilibrium under transverse load. Uniqueness of solution is proved for regions that do not extend to infinity in all directions. General analysis is given for the following cases: (a) a half-plane, (b) a region that admits of conformal mapping on to a half-plane, and (c) an infinite strip. Simple closed solutions are given of the concentrated load problem for (a) and (c) and an example of (b).

Reviewer's note. The analysis determines Green's functions, and general solutions are immediately deduced. Related problems of wider physical interest are therefore easily treated.
H. G. Hopkins (Sevenoaks).

Matildi, Pietro. Una soluzione della lastra rettangolare incastrata. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **89**, 140–157 (1955).

This paper concerns the determination of approximate solutions to the general problem of a thin, rectangular, isotropic, elastic plate in equilibrium under an arbitrary transverse load. The plate is assumed built-in at its edges. In standard notation the differential equation of equilibrium is $\nabla^4 w - q/D = 0$ and the boundary conditions are $w = \partial w/\partial n = 0$. The method of least mean-square error is adopted, i.e. the functional

$$J[w_n] = \int_S (\nabla^4 w_n - q/D)^2 dS$$

is minimised for selected test functions w_n satisfying the boundary conditions [see L. Collatz, Numerische Behandlung von Differentialgleichungen, Springer, Berlin, 1951, pp. 321 ff.; MR **13**, 285]. Here w is approximated by polynomials in rectangular cartesian coordinates. The method is illustrated by the derivation of approximate solutions to some simple problems.
H. G. Hopkins (Sevenoaks).

Alexander, J. M., and Ford, H. On expanding a hole from zero radius in a thin infinite plate. Proc. Roy. Soc. London. Ser. A. **226**, 543–561 (1954).

In common with previous theories [G. I. Taylor, Quart. J. Mech. Appl. Math. **1**, 103–124 (1948); MR **10**, 83; R. Hill, The mathematical theory of plasticity, Oxford, 1950, pp. 307–313; MR **12**, 303] the theory set out in the paper assumes a uniform hydrostatic pressure expanding a hole in a plate and also that the axial stress perpendicular to the middle surface of the plate is zero. Moreover, it is assumed that the plate varies in thickness proportionally with the radius; the problem is then one of plane stress in which all variables become functions of a single parameter r/c, where r is any radius and c is the radius of the current plastic-elastic interface. The equilibrium equation, the compressibility relation and incremental type of stress-strain relations are expressed in terms of this parameter and of the velocity v of any element, using c as the time scale. The resulting four equations are solved by taking finite increments of the variables. The residual stresses on release of the expanding pressure are determined assuming no secondary yielding occurs.

The stress and displacement patterns obtained in the above described manner are compared with those for a plate of constant thickness (Taylor, Hill). It is to be noted that Prager's treatment of the same problem [J. Appl. Mech. **20**, 317–320 (1953); MR **15**, 583] offers closed-form solutions. The comparisons of Prager's solutions with the results of the present paper would have been interesting.
E. T. Onat (Ankara).

Wilkes, E. W. On the stability of a circular tube under end thrust. Quart. J. Mech. Appl. Math. **8**, 88–100 (1955).

In all of the large literature on special problems of elastic stability, one or another particular non-linear elastic law is assumed. Recent work on finite elastic strain shows that these laws are unreliable in simple cases where an exact general solution is possible. Thus it is natural to question all existing work on stability and to seek mathematically exact criteria based on the general theory. The present paper is noteworthy in being the first to make an attempt of this kind, even though the success achieved is limited.

The author considers Southwell's problem of the stability of a circular tube subject initially to a finite longitudina- extension of any magnitude. He employs the theory of small deformation superposed on finite deformation of an incoml pressible material with arbitrary strain energy as worked out by Green, Rivlin, and Shield [Proc. Roy. Soc. London. Ser. A. **211**, 128–154 (1952); MR **13**, 884]. The author considers only infinitesimal stability with respect to deformations $f \cos \nu z \cos \kappa\theta$ whose amplitudes f, determined by the differential equations, are elaborate combinations of Bessel functions. As a condition that the assumed deformation be possible when the curved surfaces are free of traction, the author obtains the vanishing of an elaborate determinant of sixth order. Instability is said to set in at the least extension compatible with a deformation of this class. There are two free parameters, ν and κ, at least one of which must be varied in order to vary the corresponding extension. In all his further work the author sets $\kappa = 0$, thus restricting attention to symmetrical buckling. In no case does he discuss the number of roots of the determinantal equation but always proceeds as if there were only one.

For his first example the author develops the determinant in powers of the thickness of the tube, and neglects all but the lowest significant power. The buckling load then agrees with that calculated by Southwell from a special elastic law [Philos. Trans. Roy. Soc. London. Ser. A. **213**, 187–244 (1913)]. This load is known to be higher than is observed in experimental test. [This failure of agreement is no reflection on the theory of elasticity, since both Southwell and the author have added very restrictive (though different) assumptions.] The author derives also numerical estimates for thick tubes and for the solid cylinder, but only in the case of a particular elastic law.

To the reviewer it seems unfortunate that the results on this major problem have been published at such an inconclusive stage of the research. Also it would be useful to consider here the stability criterion of Hadamard [§§269–271 of Leçons sur la propagation des ondes, Hermann, Paris, 1903].
C. Truesdell (Bloomington, Ind.).

Bondar, M. G. Electrical modelling of dynamical stability of systems of rods. Dopovidi Akad. Nauk Ukrain. RSR **1954**, 356–360 (1954). (Ukrainian. Russian summary)

Denisyuk, I. N. Nomograms for determination of the loss of strength of a shaft cable according to G. N. Savin's method. Ukrain. Mat. Ž. 7, 96–100 (3 inserts) (1955). (Russian)

Colombo, Giuseppe. Sopra un problema della dinamica del binario. Rend. Sem. Mat. Univ. Padova 24, 230–244 (1955).

Although the physical system is idealized considerably, this is in effect a study of the waves generated in an elastic roadbed by a four-wheeled vehicle proceeding at constant speed. The problem is not new, but the present treatment marks an advance in that it takes account of various features of the physical situation which have been neglected heretofore. For certain critical values of the speed, determined by a characteristic equation, the amplitude of the waves is large. The author obtains various theorems concerning these critical speeds. In particular, he studies the effects upon the critical speeds of the spacing between the axles of the vehicle and the distribution of the weight on the axles.

L. A. MacColl (New York, N. Y.).

Uematu, Tokio, and Kawai, Ryôzi. Approximate solution of the forced vibration of circular rings by generalized co-ordinates. Tech. Rep. Osaka Univ. 4, 265–274 (1954).

The following general assumptions are made: (i) displacements are in the plane of the ring; (ii) length of center line of rod stays constant; (iii) cross section of rod is small relative to radius of the ring; (iv) material and cross-sectional shape are uniform; (v) radial displacement is along a principal axis of inertia of the cross section. The radial and tangential components of displacement are assumed to be in the form of certain Fourier series. Kinetic and potential energy are calculated and Lagrange's equations are applied with the above Fourier coefficients regarded as generalized coordinates. Formal solutions and some numerical examples are given for various conditions of loading.

H. D. Block (Ithaca, N. Y.).

Uematu, Tokio, and Morikawa, Yoshinobu. Approximate method by generalized co-ordinates in the problem of the circular ring of stepwise varying cross-section. Tech. Rep. Osaka Univ. 4, 275–284 (1954).

The technique described in the previous review is applied to the case in which assumption (iv) is weakened as indicated in the present title. This causes some difficulties in the computations which are handled by making certain approximations. Formal solutions are obtained; also results of numerical calculations are given and compared with experimental data. *H. D. Block* (Ithaca, N. Y.).

Satô, Yasuo. Analysis of dispersed surface waves by means of Fourier transform. I. Bull. Earthquake Res. Inst. Tokyo 33, 33–48 (1955). (Japanese summary)

Barenblatt, G. I. On propagation of instantaneous disturbances in a medium with nonlinear dependence of the stresses on the strains. Prikl. Mat. Meh. 17, 455–460 (1953). (Russian)

The author studies discontinuous solutions of one-dimensional equations of elasticity, according to which tension is an essentially arbitrary function of extension. For various types of tension-extension relations, he obtains solutions with the displacement u continuous everywhere, with $u=0$ outside a finite x interval whose length increases linearly with time, $\partial u/\partial t$ and $\partial u/\partial x$ being discontinuous at one end point. *J. L. Ericksen* (Washington, D. C.).

Rahmatulin, H. A. On the propagation of plane waves in an elastic medium with nonlinear dependence of the stress on the strain. Moskov. Gos. Univ. Uč. Zap. 152, Meh. 3, 47–55 (1951). (Russian)

In linear elasticity, velocity waves and all weaker types of discontinuity surfaces must be characteristics of the equations of motion. In nonlinear elasticity, weaker types of discontinuities continue to be characteristics, but there is reason to expect velocity waves not to be. One thing which suggests this is the known distinction between shock waves and Mach waves in ideal gases.

The author investigates velocity waves and characteristic surfaces in nonlinear elasticity, assuming a one-dimensional situation where tension is an essentially arbitrary function of extension. He points out that, in general, in a material region containing the wave, it is impossible for the rate of change of total energy to be balanced by the rate at which the stresses do work. Here total energy means strain energy plus kinetic energy. This suggests that these waves involve processes which are neither adiabatic nor isothermal. He determines the characteristics and discusses some of their properties. *J. L. Ericksen* (Washington, D. C.).

Nowiński, Jerzy. Basic principles of the theory of plasticity. I. Seven lectures. Rozprawy Inż. 2, 69–141 (1954). (Polish. Russian and English summaries)

An elementary introduction to the theory of plasticity is presented in six out of the seven lectures, limited to a discussion of states of stress and strain, plasticity condition and related Hencky stress-strain relations, and the solution of the elastic-plastic problems of the thick-walled tube under radial pressure and the beam in pure bending. The seventh lecture presents some elementary concepts of crystal physics referring to slip and mechanical strength. No notice is taken of the work in plasticity in the U. S. A. and in England during the last twenty years and no reference at all is made to the theory of glide-line fields and the resulting methods of integration of plasticity problems.

A. M. Freudenthal (New York, N. Y.).

Galin, G. Ya. On conditions on surfaces of strong discontinuities for elastic and plastic bodies. Prikl. Mat. Meh. 19, 368–370 (1955). (Russian)

A brief thermodynamical discussion is given of a shock in plastic material. The usual three equations of continuity, momentum and energy are first stated. The condition of increase in entropy is formulated, and is discussed for a simple case.

H. G. Hopkins (Sevenoaks).

Küssner, H. G. Theorie des elastisch-plastischen Kontinuums mit freier Oberfläche bei veränderlicher Belastung. Z. Angew. Math. Mech. 35, 130–143 (1955). (English, French and Russian summaries)

The author is concerned mainly with materials for which the stress is a linear function of the infinitesimal strain tensor and its time derivatives. More general constitutive equations are mentioned. It should be noted that his equation (19a), which is intended to hold for finite deformations, does not satisfy the very reasonable invariance requirements imposed by Cotter and Rivlin [Quart. Appl. Math. 13, 177–182 (1955); MR 16, 1067] in their investigation of equations of this type. For the linear equations, the author derives rather general solutions for longitudinal and transverse

waves. Using these, he obtains solutions for the half-space bounded by a plane, the plane boundary being subject to specified normal reactions. Gravity is taken into account. The case where the tractions are applied to an elastic plate in contact with the plane boundary is discussed.

J. L. Ericksen (Washington, D. C.).

Kačanov, L. M. On compound loading. Prikl. Mat. Meh. 19, 371–375 (1955). (Russian)

Attention is given to states of homogeneous stress and strain-rate for material obeying the Prandtl-Reuss equations. Discussion is given of certain simple situations in which limiting states of stress exist for prescribed deformations which are indefinitely enforced. The analysis hinges on certain results concerning the asymptotic behaviour of a general class of first order, ordinary, non-linear differential equations.

H. G. Hopkins (Sevenoaks).

Moskvitin, V. V. Residual stresses and strains in a hollow thick-walled sphere. Vestnik Moskov. Univ. 7, no. 8, 57–61 (1952). (Russian)

A hollow thick-walled sphere of material is plastically stressed under slowly-increasing uniform internal pressure. Analysis based upon total-strain theory is given to determine the state of residual stress and strain following removal of the pressure. Numerical results are given for a typical example.

H. G. Hopkins (Sevenoaks).

Huber, Maksymilian T. Some remarks on mechanical properties of solid bodies. Arch. Mech. Stos. 5, 5–14 (1953). (Polish)

The paper represents a fragment of a lecture of the author given in 1950 and has been published by the Polish Academy of Sciences in honor of his memory. It discusses the discrepancy between the atomic and the technical cohesive strength and the size effect in fracture tests with particular reference to the Griffith theory.

A. M. Freudenthal.

MATHEMATICAL PHYSICS

Bastin, E. W., and Kilmister, C. W. The concept of order. II. Measurements. Proc. Cambridge Philos. Soc. 51, 454–468 (1955).

[For part I see same Proc. 50, 278–286 (1954); MR 15, 760.] From the idea of the continuous development of an investigation, as expressed in the use of indefinitely continuing sequences of procedures, the concept of a measurement together with its result is formulated. The resulting theory is initially very general, but it is possible to restrict it in such a way as to make the construction of a consistent world possible. The physical condition providing the restrictions that make this construction possible is found in the use of the general idea of a test-particle in all fundamental investigations, which is shown to be a case of the use of the theory-languages of the previous paper. Finally, the theory is applied to the solution of the problem of preferred inertial frames. (Authors' summary.)

A. H. Taub (Urbana, Ill.).

***Kustaanheimo, Paul. On some special functions in Galois fields.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 169–175 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

A model of physics in which space is discrete and finite while time has the order type of the integers is considered. Space is taken to be the three-dimensional cartesian geometry over the p-element field F where p, a very large prime, is so chosen that the order relation, $a>b$ if and only if $a-b$ is a quadratic residue, behaves nearly like the ordinary ordering of the rational field. The motion of a physical system is assumed to be given by a finite recursion. The author observes that the solutions of such a recursion must be "periodic with a primitive period not divisible by p^2". He seems to have overlooked two facts. The solution need only be eventually periodic (as the decimal expansion of $5/12=.41666\cdots$) and the non-repetitive parts can be arbitrarily long. The recursion $x_{n+1}=x_n{}^p$ has primitive period p^2 if x_0 is chosen in the p^2-th degree extension of F but not in the pth degree extension. Finally he offers a connection between his model and the disappearance and creation of matter in the universe.

A. M. Gleason.

Optics, Electromagnetic Theory

Hopkins, H. H. The frequency response of a defocused optical system. Proc. Roy. Soc. London. Ser. A. 231, 91–103 (1955).

An analytic study is made of the effect of defocusing on the response of an aberration-free optical system to line frequencies in the object. It is shown that all the information about the image-forming properties of any system may be obtained by means of unidimensional objects. Curves are given showing the response as a function of line-frequency for various amounts of defect of focus. Comparison is made with the results to be expected from geometrical optics. A tolerance for defect of focus is derived and appears to agree well with experimental results. Both circular and rectangular apertures are discussed.

E. W. Marchand.

Schiller, Ralph. New transition to ray optics. Phys. Rev. (2) 97, 1421–1428 (1955).

A. Sommerfeld and J. Runge, using a suggestion of P. Debye, showed how the eikonal equation of geometrical optics can be derived from the scalar wave equation in the limiting case of vanishingly short wave length [Ann. Physik (4) 35, 277–298 (1911)]. In more recent times the corresponding transition from the vector wave equation was investigated by a number of authors but no adequate attention appears to have been paid to the fact that several phase functions (one for each Cartesian component of the field vectors or of the vector potential) have to be introduced, leading to a number of distinct eikonal functions. The present paper is concerned with this point and discusses one possible representation of the geometrical-optics field associated with the vector field of the electromagnetic theory.

In this representation use is made of the fact that the energy flow is not defined uniquely by the field vectors. It is defined here by the expression

$$\mathbf{J}=\text{const.}\times[\nabla\mathbf{A}^*)\cdot\mathbf{A}-(\nabla\mathbf{A})\cdot\mathbf{A}^*],$$

where $\mathbf{A}$ is the (monochromatic) complex vector potential. $\mathbf{J}$ differs from the time-averaged Poynting vector by a term proportional to $\nabla\times(\mathbf{A}^*\times\mathbf{A})$, and in consequence the associated velocity (ray) field differs from the usual one by terms which are interpreted as the "intrinsic vorticity" of the

medium. A generalized eikonal equation which involves the phase functions and certain associated parameters analogous to the hydrodynamical parameters of Clebsch is derived.

E. Wolf (Manchester).

✶Sturrock, P. A. Static and dynamic electron optics. An account of focusing in lens, deflector and accelerator. Cambridge, at the University Press, 1955. x+240 pp. $5.50.

The number of books on electron optics available in English is quite limited, and there is a great need of a treatment emphasizing the mathematical techniques. The Table of contents of Sturrock's book promises to remedy this need even though the book is restricted entirely to the ray approach to the subject as opposed to the ballistical approach. In section II on "Classical geometrical optics" it lists such chapters as Hamilton's characteristic functions, The Lagrange invariant, Perturbation characteristic functions, The reciprocal and imaging relations, Normal congruences, and The Hamilton-Jacobi equation. The approach used in the treatment of periodic focusing in particle accelerators is shown by the inclusion of such chapters as Perturbation calculations by variation of parameters and Perturbation calculations by matrix methods. But while the choice of topics is excellent, the presentation is such as to discourage many readers. The text appears to be beyond most students of electron optics no matter how excellent their mathematical background. The notation alone makes the book impractical for occasional reference by a designer of electron-optical devices. Another objectionable feature is the use of electron-optical units with e.g. the unit of current set at 1356 amperes. As far as minor corrections are concerned, it would have been helpful to have the term "eiconal", which is in fairly standard usage, included and defined in the text to permit cross-reference to other texts. As far as this reviewer can judge, Sturrock's book will be appreciated only by a reader—well versed in the subject matter—interested in finding a formal and rigorous derivation of the concepts and results of electron optics. All others need a translation.

J. E. Rosenthal (Passaic, N. J.).

Bonštedt, B. E. A method of finding a wide class of electrostatic and magnetic fields for which the solutions of the basic equation of electron optics are expressed by means of known functions. Ž. Tehn. Fiz. **25**, 541–543 (1955). (Russian)

This method of finding the conditions under which the paraxial ray equation can be readily solved is based on the use of invariants of ordinary linear second-order differential equations and of the Schwarzian derivative [cf., e. g., E. L. Ince, Ordinary differential equations, p. 394, Longmans-Green, London, 1927]. The invariant of the paraxial ray equation is

$$I_r(x)=\tfrac{3}{16}(\varphi_x'/\varphi)^2+\tfrac{1}{8}(e/m)H^2(x)/\varphi(x),$$

where $\varphi(x)$ is the electrostatic potential and $H(x)$ the magnetic field intensity. Given two ordinary linear second-order differential equations whose solutions are known, the method shows: 1. How to use these solutions to determine a function which will be a solution of a paraxial ray equation. 2. How to determine $\varphi(x)$ and $H(x)$ from the invariants of the auxiliary equations. Examples are given where the solutions of the two auxiliary equations are Bessel functions of different orders and also where they are Legendre functions of different orders. (A misprint occurs in Eq. (4) whose first term should read $\tfrac{1}{2}\xi_x'''/\xi_x'$.)

J. E. Rosenthal.

Burger, A. P. On the asymptotic solution of wave propagation and oscillation problems. Thesis, Technische Hogeschool te Delft, 1955. Nationaal Luchtvaartlaboratorium, Amsterdam, Rep. F. 157, 97 pp. (1954).

A major problem in diffraction theory is concerned with a study of the transmission cross-section of a screen with a slit or a strip, when the product of the slit or strip width a by the wave number k is large. These two problems which are essentially complimentary can be solved by expansion of field quantities in a series of functions which are natural to the problem, but unfortunately, the transmission cross-section can only be computed for ka small.

Following a suggestion of R. Timman, the author studies the solutions of the time-varying wave equation, rather than the equation in the steady state, the Laplace transform of the time-dependent equation being basically the steady-state equation. The advantage of this procedure, is that one has a hyperbolic equation, rather than an elliptic equation to consider. This enables the author to apply the theory of characteristics. The problem is equivalent to a semi-infinite plate in stationary supersonic flow under small angle of attack, and this reduces the problem to one in lifting-surface theory. Thus the author can find the deviation from the Kirchhoff approximation for the case of a slit in an infinite plane. Other examples are considered. The methods and calculations are too detailed to describe in full.

A. Heins (Pittsburgh, Pa.).

Franz, Walter, und Galle, Raimund. Semiasymptotische Reihen für die Beugung einer ebenen Welle am Zylinder. Z. Naturf. **10a**, 374–378 (1955).

In a previous paper by the first author [same Z. **9a**, 705–716 (1954); MR **16**, 764], Watson's transformation was applied to the Green's functions of the cylinder and the sphere with the purpose of obtaining asymptotic formula describing the correction terms to the geometrical-optics values. The same procedure is applied in this paper to the problem of the title. A numerical discussion is included.

C. J. Bouwkamp (Eindhoven).

Nomura, Yûkichi, and Katsura, Shigetoshi. Diffraction of electromagnetic waves by circular plate and circular hole. J. Phys. Soc. Japan **10**, 285–304 (1955).

The incident wave is plane polarized and hits the obstacle at an arbitrary angle. The two problems of the title are treated simultaneously in two columns [instead of applying Babinet's principle]. Fields are derived from a two-component Hertzian vector. Boundary and edge conditions lead to a system of linear equations in an infinite number of unknowns. The method applied is that of King, Sommerfeld, and others, using integrals over Bessel functions resulting in hypergeometric polynomials. Various diagrams illustrate the authors' numerical results. Some [not all] numerical results by other authors for the transmission cross-section of the circular hole at normal incidence are displayed and compared with their own results.

C. J. Bouwkamp.

Schensted, Craige E. Electromagnetic and acoustic scattering by a semi-infinite body of revolution. J. Appl. Phys. **26**, 306–308 (1955).

The theory of M. Kline [Comm. Pure Appl. Math. **4**, 225–262 (1951); MR **13**, 408] of asymptotic solution to hyperbolic partial differential equations for short wavelengths is applied to the scattering of a plane (electromagnetic or acoustic) wave incident along the axis of a perfectly reflecting semi-infinite body of revolution. Two terms of

Kline's expansion are obtained, of which the first is the geometric-optics solution. This first term is identical with the exact solution in the case of a paraboloid of revolution. A comparison of the asymptotic and physical-optics (current-distribution approximation) solutions is included.
C. J. Bouwkamp (Eindhoven).

Siegel, K. M., Crispin, J. W., and Schensted, C. E. Electromagnetic and acoustical scattering from a semi-infinite cone. J. Appl. Phys. **26**, 309–313 (1955).

The value of the nose-on back-scattering cross-section is determined from the exact solution of Hansen and Schiff (unpublished reports of Stanford University) by applying Euler summation to the slowly convergent series. As may be expected from simple physical reasoning, it turns out that the physical-optics approximation is almost identical with the exact solution for all finite wavelengths. Experimental results are compared with exact and physical-optics predictions. *C. J. Bouwkamp* (Eindhoven).

Wait, James R. Scattering of a plane wave from a circular dielectric cylinder at oblique incidence. Canad. J. Phys. **33**, 189–195 (1955).

The cylinder is infinite, homogeneous and isotropic while the magnetic vector of the incident wave is transverse to the axis. The scattered field contains a cross-polarized component which vanishes at normal incidence. Various limiting cases in regard to the physical parameters are considered. The cylinder may be considered as an idealized model for a meteor trail. Numerical results are to be reported in a subsequent paper. *C. J. Bouwkamp* (Eindhoven).

Wait, James R. Scattering of electromagnetic waves from a "lossy" strip on a conducting plane. Canad. J. Phys. **33**, 383–390 (1955).

The two-dimensional problem of a line source of current over a conducting plane with a lossy strip is solved in two different ways. In the first, which is an approximation used in various practical problems, the effect of the conducting plane is thrown into a surface-impedance inhomogeneous boundary condition, so that the field in free space is obtained at once by Green's formula. In the second, exact, method the author applies elliptic-cylinder coordinates and expansions in Mathieu functions. Some numerical results show the accuracy of the approximation compared with the exact theory. *C. J. Bouwkamp* (Eindhoven).

Williams, W. Elwyn. Step discontinuities in waveguides. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-77**, i+32 pp. (1955).

The problem of a step-like discontinuity in a rectangular wave guide has been previously considered by the so-called static method or a Fourier series method [N. Marcuvitz, Waveguide handbook, McGraw-Hill, New York, 1951]. Here the author shows that by using a longitudinal matching surface, such problems may be reduced to a system of infinitely many equations in infinitely many unknowns. The interesting physical parameters may be expressed in terms of the known solution of a related problem (the bifurcated guide) plus corrections from this system which are small. Other problems of this type may be treated by this device. The methods employed are a variation of the methods of Wiener and Hopf which bring forth the system of equations.
A. E. Heins (Pittsburgh, Pa.).

Kuznecov, P. I., and Stratonovič, R. L. Electromagnetic processes in a two-conductor system. Èlektričestvo **1955**, no. 2, 5–13 (1955). (Russian)

The authors consider the problem of propagation of electromagnetic waves along a parallel wire line. The wires are of the same material and of the same diameter. The wavelength within the wire medium, λ_i, is assumed small compared to that in the external medium, λ_e, and the latter is assumed to be much greater than the spacing between wires. No assumption is made with respect to the thickness of the wire, a. The propagation constant of the dominant mode is calculated. The method yields essentially a power series in (λ_i/a). The first term is shown; the method of obtaining further terms is indicated. The series converges rapidly for all ratios of a to a wire spacing. A generalization to wires of unequal diameter is also given. *J. Shmoys.*

Lüst, R. Plasmaschwingungen in einem äusseren Magnetfeld. Z. Astrophys. **37**, 67–71 (1955).

Bekannterweise ist die gestörte Komponente der Radiostrahlung der Sonne so gross, dass es unmöglich erscheint diese als thermische Strahlung (also mit Hilfe der Planckschen Strahlungsformel) zu deuten. Bei den bekannten Langmuirschen Plasmaschwingungen, die longitudinale Wellen sind und gewissermassen als entartete longitudinale elektromagnetische Wellen aufgefasst werden können, verschwindet dagegen der Poyntingsche Vektor, es tritt also keine Ausstrahlung auf. Der Verfasser zeigt jedoch, dass die Vehältnisse ganz anders werden, wenn eine ebene Welle sich senkrecht zu einem Magnetfeld im Plasma ausbreitet und die Schwingungsrichtung senkrecht zu diesem Felde orientiert ist. Diese Welle ist dann weder rein transversal noch rein longitudinal und strahlt infolge ihres transversalen Anteiles (da der Poyntingsche Vektor jetzt nicht mehr verschwindet) Energie aus, welche also zur Deutung der erwähnten Radiostrahlung herangezogen werden könnte.

Den Ausgangspunkt der Rechnungen bildet die Gleichung der Impulsbilanz des Plasmas

$$(1) \qquad \rho\frac{\partial \mathfrak{v}}{\partial t}=\frac{1}{c}[\mathfrak{j}\mathfrak{H}_0],$$

wo ρ die Dichte, $\mathfrak{v}$ Die makroskopische Geschwindigkeit, $\mathfrak{j}$ die wahre Stromdichte und $\mathfrak{H}_0$ das äussere Magnetfeld bedeuten. Zu dieser Gleichung kommt noch die Diffusionsgleichung des Plasmas

$$(2) \qquad \frac{4\pi}{\omega_p^2}\left(\frac{\partial \mathfrak{j}}{\partial t}+\gamma \mathfrak{j}\right)=\mathfrak{E}+\left[\frac{\mathfrak{v}}{c}\mathfrak{H}_0\right]-(m_i-m_e)\frac{1}{e}\frac{\partial \mathfrak{v}}{\partial t},$$

wo $\mathfrak{E}$ die elektrische Feldintensität bedeutet und

$$\omega_p^2=4\pi e^2N/m_e$$

ist. Die übrigen Symbole haben alle die gewohnte Bedeutung. Zu (1) und (2) werden noch die Maxwellschen Gleichungen hinzugenommen und daraus werden die erwähnten Resultate hergeleitet. *T. Neugebauer* (Budapest).

Chenon, René. Théorie classique des champs. C. R. Acad. Sci. Paris **241**, 166–167 (1955).

Aržanyh, I. S. Representation of an electromagnetic field by retarded potentials. Dokl. Akad. Nauk SSSR (N.S.) **100**, 1053–1056 (1955). (Russian)

In the present note the author arrives at certain formulas in the theory of vector fields which are the natural extension to the dynamic case of formulas derived earlier [same Dokl. (N.S.) **85**, 55–58 (1952); MR **14**, 46] for the static case. All

the formulas obtained are applications of an integral representation formula for a vector field $V(x_1, x_2, x_3, t)$ which satisfies the pair of equations rot $V=\Omega$, div $V=\theta$. Two applications are made, one to boundary-value problems considered in the earlier note by means of the operational calculus, and to the equations of motion of an electron gas in a vacuum.

J. B. Dias (College Park, Md.).

Kaufman, R. N. A dielectric layer with a spherical cavity in a homogeneous electrostatic field. Dokl. Akad. Nauk SSSR (N.S.) **101**, 633–636 (1955). (Russian)

This paper presents an exercise on the solution of boundary-value problems using expansions in terms of Legendre functions. The problem is more restricted than stated in the title. It deals with a plane dielectric layer of infinite extent parallel to the direction of the electric field with a spherical cavity equidistant from the boundary planes. A brief statement is made concerning the case where the center of the cavity is at any point in the dielectric.

J. E. Rosenthal (Passaic, N. J.).

Belevitch, Vitold. Synthèse des réseaux électriques passifs à n paires de bornes de matrice de répartition prédéterminée. Ann. Télécommun. **6**, 302–312 (1951).

The necessary and sufficient conditions that an n by n matrix Z be the impedance matrix of a network with n pairs of terminals have been obtained by Oono, Bayard, Leroy, McMillan, and Tellegen. A new approach to this problem is given in this paper with the introduction of the matrix $S=(Z+I)^{-1}(Z-I)$ termed the scattering matrix. A first advantage is that S can be defined for degenerate networks for which the impedance or admittance matrices cannot be defined. Necessary and sufficient conditions are found for a matrix to be the scattering matrix of a network. An n-pair network can be achieved by terminating p of the pairs of an $(n+p)$-pair reactive network in p resistors. Networks of this type are characterized.

R. J. Duffin.

Oono, Yosiro, et Yasuura, Kamenosuke. Synthèse des réseaux passifs à n paires de bornes donnés par leurs matrices de répartition. Ann. Télécommun. **9**, 73–80, 109–115, 133–140 (1954).

The first synthesis of a network to realize a given impedance matrix Z was given by Oono [J. Inst. Elec. Commun. Engrs. Japan **29**, 82–87 (1946); J. Math. Phys. **29**, 13–26 (1950); MR **12**, 567]. A new approach to the synthesis problem was given by V. Belevitch in the paper reviewed above by the introduction of the scattering matrix $S=(Z+I)^{-1}(Z-I)$. This paper also treats the synthesis problem based on properties of the scattering matrix but the treatment differs from that of Belevitch. The authors determine the minimal number of network elements needed in the synthesis; this problem has also been analyzed by B. McMillan [Bell System Tech. J. **31**, 217–279, 541–600 (1952); MR **14**, 116] and B. D. H. Tellegen [J. Math. Phys. **32**, 1–18 (1953); **15**, 377]. The last section of the paper is devoted to networks which violate the principle of reciprocity. Such networks result if gyrators are employed in addition to conventional network elements. A gyrator has a skew symmetric impedance matrix. The synthesis problem is treated for such non-reciprocal networks.

R. J. Duffin (Pittsburgh, Pa.).

Fialkow, Aaron, and Gerst, Irving. The transfer function of networks without mutual reactance. Quart. Appl. Math. **12**, 117–131 (1954).

The ratio of output to input voltages of a network as a function of the complex frequency p is termed the transfer ratio $A(p)$. A complete theory of $A(p)$ is developed for networks containing resistance, capacitance, and self-inductance, but no mutual inductance. They first consider the network with three external terminals and write $A(p)=KN(p)/D(p)$ when N and D are polynomials in p and have leading coefficient 1. The necessary and sufficient condition on $A(p)$ may be described as follows: The poles of $A(p)$ are in the left half-plane or on its boundary, except that $p=0$ and $p=\infty$ are excluded. A pole on the imaginary axis must be simple and have a pure imaginary residue. The zeros of $A(p)$ cannot have positive real part but are otherwise arbitrary. The range of K is an interval $0<K\leqq K_0$ when K_0 is the minimum value of D/N for $0\leqq p\leqq\infty$. K_0 is a realizable value of K if and only if the minimum value is assumed only at $p=0$ or $p=\infty$ or both. Next the network with four external terminals is treated. The necessary and sufficient conditions differ from those above in that there is no restrictions on the zeros and that the range of K is now $-K_0\leqq K\leqq K_0$, where K_0 is the minimum value of D/N for $0\leqq p\leqq\infty$. Again K_0 is a realizable value of K if and only if the minimum value is assumed at $p=0$ or $p=\infty$ or both. The techniques of analysis and synthesis are extensions of those employed previously by the authors to treat networks containing only resistance and capacitance [same Quart. **10**, 113–127 (1952); MR **14**, 116]. *R. J. Duffin.*

Brachman, Malcolm K. Note on the Kramers-Kronig relations. J. Appl. Phys. **26**, 497–498 (1955).

Some of the well known integral formulae of Bode relating network functions are derived by use of the Mellin transform. *R. J. Duffin* (Pittsburgh, Pa.).

Sternberg, Robert L. A general solution of the two-frequency modulation product problem. III. Rectifiers and limiters. J. Math. Phys. **33**, 199–205 (1954).

This paper analyzes rectifiers and limiters by methods developed in a previous paper of the same title by R. L. Sternberg and H. Kaufman [same J. **32**, 233–242 (1954); MR **15**, 832]. *R. J. Duffin* (Pittsburgh, Pa.).

Carlin, H. J. On the physical realizability of linear non-reciprocal networks. Proc. I. R. E. **43**, 608–616 (1955).

The properties of the impedance matrix and the scattering matrix for a non-reciprocal network (fixed frequency) are analyzed. A canonical representation of such networks is given by employing gyrators in addition to conventional circuit elements. *R. J. Duffin* (Pittsburgh, Pa.).

Storch, Leo. Synthesis of constant-time-delay ladder networks using Bessel polynomials. Proc. I. R. E. **42**, 1666–1675 (1954).

The mathematical problem in delay network design is the approximation of the exponential function by rational functions. In this paper the approximation is achieved by the Bessel polynomials of Krall and Frink [Trans. Amer. Math. Soc. **65**, 100–115 (1949); MR **10**, 453]. The network realization is of the ladder type. The resulting transfer function is low-pass, minimum-phase, and maximally flat. The response to an impulse resembles a Gaussian curve properly delayed. A desirable feature of this realization is that losses are permitted in the reactances. *R. J. Duffin.*

∗Kegel, Günter. On the representation of orthogonal polynomials by transmission lines (preliminary note). New research techniques in physics, pp. 395–400. Symposium organized by the Academia Brasileira de Ciências and Centro de Cooperación Científica para America Latina (UNESCO) under the auspices of the Conselho Nacional de Pesquisas do Brasil, Rio de Janeiro and São Paulo, July 15–29, 1952. Rio de Janeiro, 1954.

Certain classes of polynomials satisfy three-term recursion formulae; the Legendre polynomials are an example. It is observed that the potentials from section to section of a lumped line satisfy corresponding formulae. It is proposed to obtain a representation of polynomials by such networks. The variable would be the frequency and the degree would be related to the number of meshes. *R. J. Duffin.*

Ozaki, Hiroshi, and Fujisawa, Toshio. Approximation problems in RC network synthesis. Tech. Rep. Osaka Univ. **3**, 243–248 (1953).

Of concern is the r.m.s. approximation of a derived characteristic by a function with prescribed simple poles on the negative real axis. Appeal is made to the closure theorem of Szász. An example is given of an approximation of a given phase and amplitude characteristic for a finite range of frequency. *R. J. Duffin* (Pittsburgh, Pa.).

Quantum Mechanics

∗de Broglie, Louis. Eléments de théorie des quanta et de mécanique ondulatoire. Traité de physique théorique et de physique mathématique, tome III. Gauthier-Villars, Paris, 1953. viii+302 pp. 2880 francs.

This book presents essentially the course that the author has been giving at the Ecole Normale Supérieure since 1934. The chapter headings cover the following topics: a summary of the Maxwell theory and the Lorentz electron theory, the special theory of relativity, classical statistical mechanics, black-body radiation, the particle theory of light, the Bohr-Sommerfeld theory of the atom, the correspondence principle, the basis and physical significance of wave mechanics and its applications to quantization, Heisenberg mechanics, the probabilistic interpretation of wave mechanics, the spin of the electron and the Dirac theory, the Pauli principle and the wave mechanics of a system of particles, quantum statistics. As these topics indicate, the development of the quantum theory is presented essentially along historical lines, beginning with Planck's law of black-body radiation and Einstein's law of the photoelectric effect. In general, the emphasis is on the physical aspects rather than on the formal and on the foundations of the theory rather than on its applications. The experimental bases of various concepts of the quantum theory are discussed. On the other hand, mathematical results are often presented without proof. A short bibliography, chiefly of books in French, is given at the end of each chapter. *N. Rosen* (Haifa).

∗de Broglie, Louis. La physique quantique, restera-t-elle indéterministe? Gauthier-Villars, Paris, 1953. vii+113 pp. 1500 francs.

The first part summarizes the author's ideas on the interpretation of the quantum theory and the changes they have undergone in the course of time. The second part reproduces a number of early papers of the author dealing with questions of the interpretation of quantum theory [C. R. Acad. Sci. Paris **179**, 1039–1041 (1924); **183**, 447–448 (1926); **184**, 273–274; **185**, 1118–1119 (1927); J. Phys. Radium (6) **8**, 225–241 (1927) (with some added comments by the author)]. The third part reproduces a number of recent papers by the author [C. R. Acad. Sci. Paris **233**, 641–644 (1951); **234**, 265–268; **235**, 557–560, 1345–1349, 1453–1455 (1952); MR **14**, 117, 435, 520] and two by J.-P. Vigier [ibid. **235**, 1107–1109, 1372–1375 (1952); MR **14**, 436]. In addition there appears a work by J.-P. Vigier, "Physique relativiste et physique quantique" (pp. 89–111) on a generalization of the Einstein relativity theory in which singularities, representing particles, are accompanied by waves described by functions satisfying equations similar to those of the quantum theory. *N. Rosen* (Haifa).

∗Pauli, W. Remarques sur le problème des paramètres cachés dans la mécanique quantique et sur la théorie de l'onde pilote. Louis de Broglie, physicien et penseur, pp. 33–42. Editions Albin Michel, Paris, 1953. 870 francs.

The author criticizes the work of L. de Broglie [see references in the preceding review] and of D. Bohm [Phys. Rev. (2) **85**, 166–179, 180–193 (1952); MR **13**, 709, 710] on a deterministic interpretation of wave mechanics by means of hidden parameters. *N. Rosen* (Haifa).

∗Destouches, Jean-Louis. Retour sur le passé. Louis de Broglie, physicien et penseur, pp. 67–85. Editions Albin Michel, Paris, 1953. 870 francs.

This article discusses, among other things, some aspects of the process of measurement of a physical system and the deterministic interpretation of the quantum theory [see the preceding review]. *N. Rosen* (Haifa).

∗Rosenfeld, L. L'evidence de la complémentarité. Louis de Broglie, physicien et penseur, pp. 43–65. Editions Albin Michel, Paris, 1953. 870 francs.

The complementarity principle is discussed in the light of experience, and its relation to various interpretations of the quantum theory is considered. *N. Rosen* (Haifa).

Watanabe, Satosi. Symmetry of physical laws. III. Prediction and retrodiction. Rev. Mod. Phys. **27**, 179–186 (1955).

[For parts I and II see Rev. Mod. Phys. **27**, 26–39, 40–76 (1955); MR **16**, 883, 890.] The author discusses the problem of retrodiction, by which is meant statistical inference concerning the state of a system in the past on basis of present observations. The mathematical formulation of retrodiction in quantum mechanics is so presented that it shows great symmetry with the more familiar case of prediction. Attention is drawn on the physical differences between the two cases, which are of course essential despite all formal similarities. *L. Van Hove* (Utrecht).

∗Järnefelt, G. An attempt to work out a finite system corresponding to a special case of Schrödinger's nonrelativistic model of the linear harmonic oscillator. Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 116–134 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

Models of plane geometry are considered containing only those points lying in some large square of the Cartesian plane having rational coordinates and a bound on the size

of the denominators. Since physical space is neither demonstrably infinite nor demonstrably continuous, this finite geometry will be operationally indistinguishable from the usual plane as a representation of physical space, if the bounds are chosen large enough. This model can be embedded in the coordinate geometry over the p-element field F if the prime p is suitably chosen. With this in mind, the author tries to carry the Schrödinger-Kennard model of the linear oscillator over to the coordinate geometry over F. The direct attempt fails because of the non-existence of solutions of the analogues of the quantum-theoretic differential equations. However, greater success is obtained if the translation to F-valued functions is made after solving the quantum-theoretic equations. *A. M. Gleason.*

Darling, B. T., and Zilsel, P. R. The theory of finite displacement operators and fundamental length. Phys. Rev. (2) **91**, 1252–1256 (1953).

The present paper provides an improved basis for the results of an earlier work [B. T. Darling, Phys. Rev. (2) **80**, 460–466 (1950); MR **12**, 465]. From the Dirac equation for the electron an equation containing a fundamental length is obtained by replacing all the differential operators appearing there by infinite-order differential operators ("finite-displacement operators") which satisfy a certain set of postulates. It is shown that the spectrum of masses given by this equation is not affected by the presence of an electromagnetic field. The relation between the present theory and other theories is discussed. *N. Rosen.*

Mrowka, B. Zur Darstellung der Quantenmechanik. II. Relativistische Quantenmechanik, Dirac-Gleichung. Z. Physik **138**, 557–569 (1954).

In an earlier work [B. Mrowka, Z. Physik **130**, 164–173 (1951); MR **13**, 409] the author presented derivations of the non-relativistic wave equations for particles with and without spin, based on three axioms. In the present paper, by introducing a fourth axiom, that of Lorentz invariance, he derives the Pauli modification of the Dirac equation for a charged particle with arbitrary magnetic moment.
N. Rosen (Haifa).

Good, R. H., Jr. Properties of the Dirac matrices. Rev. Mod. Phys. **27**, 187–211 (1955).

The paper presents an excellent review of the mathematical properties of the Dirac matrices which are of importance for physical applications. Complete derivations are given and make use only of ordinary matrix calculus without recourse to more abstract algebraic and group-theoretical methods. Beyond the ordinary material concerning the Dirac equation and its covariance the author discusses time inversion, charge conjugation, the various special representations of the Dirac matrices, the relation to spinor calculus, as well as the construction of scalars by means of four wave functions, a topic of fundamental importance for the theory of beta-decay. *L. Van Hove* (Utrecht).

Tiomno, N. Mass reversal and the universal interaction. Nuovo Cimento (10) **1**, 226–232 (1955).

It is assumed, as an invariance principle, that the results of relativistic quantum theories should be invariant under the change in sign of the mass term in the Dirac equation. The consequences of this principle for the allowable types of interaction terms in the "Universal Fermi Interaction" among light and heavy particles are discussed.
N. Rosen (Haifa).

Phillips, R. J. N. Indefinite metrics and multi-mass field theories. Nuovo Cimento (10) **1**, 822–839 (1955).

In the first part of this paper the author reviews the definition of and the role that scalar products play in quantum theory. In particular methods of interpreting indefinite scalar products by related positive definite ones are discussed. In the second part of the paper fields satisfying differential equations of the form

$$\prod_j (\Box - k_j^2)\varphi(x) = 0 \quad \text{or} \quad \prod_j (\gamma_\mu \partial\mu + k_j)\psi(x) = 0,$$

for multi-mass fields are discussed. It is concluded that theories involving fields of this type that can be used are at least as divergent as in the single-mass formalism.
A. H. Taub (Urbana, Ill.).

Goldberger, Marvin L. Use of causality conditions in quantum theory. Phys. Rev. (2) **97**, 508–510 (1955).

Previously M. Gell-Mann, M. L. Goldberger and W. E. Thirring [Phys. Rev. (2) **95**, 1612–1627 (1954); MR **16**, 654] derived the Kramers-Kronig dispersion relations from quantum electrodynamics by imposing the causality requirement, using perturbation theory. It is shown that this result can be derived without perturbation theory, and that it is essentially independent of the form of the coupling between the electromagnetic field and the matter field.
N. Rosen (Haifa).

Costa de Beauregard, Olivier. Sur la théorie quantique de la gravitation. C. R. Acad. Sci. Paris **240**, 2383–2384 (1955).

Using the Schwinger transformation theory, the fundamental equations of the quantum theory of gravitation are formulated but only in the linearized version. *L. Infeld.*

Costa de Beauregard, O. Diffraction in time. Rev. Mexicana Fis. **3**, 185–200 (1954). (Spanish)

Spanish version of the paper translated below.

Costa de Beauregard, O. Diffraction in time. Rev. Mexicana Fis. **3**, 201–216 (1954).

In certain physical situations it is convenient, on the basis of Minkowski's space-time geometry, to make use of the symmetry between the time and space coordinates. In this way one can have a wave undergoing "diffraction in time". This is discussed on the basis of Fresnel's treatment of diffraction. The role of negative energy states is considered. A covariant theory of Fourier transforms is presented. *N. Rosen* (Haifa).

Moshinsky, Marcos. Diffraction in time associated with a distribution of sources. Rev. Mexicana Fis. **3**, 236–252 (1954). (Spanish)

By means of examples involving several types of wave equations, the author shows that the "diffraction in time" of a wave due to the opening and closing of a shutter can be described in terms of a suitable source distribution. In this way he establishes the equivalence between his approach and that of O. Costa de Beauregard [see preceding reviews]. The role of the negative energy states is discussed.
N. Rosen (Haifa).

***Pasta, John R. Limiting procedures in quantum electrodynamics.** Abridgment of a dissertation, New York University, 1951. 8 pp.

Self-mass, self-charge and self-stress of the electron are computed to the second order in a charge expansion. It is

concluded that, by suitable choice of limiting procedures, arbitrary values may be assigned to the divergent integrals in the theory. It is re-emphasized that only physical considerations can decide what values are appropriate [see Pauli and Villars, Rev. Mod. Phys. 21, 443–444 (1949); MR 11, 301]. *A. Salam* (Cambridge, England).

Gotô, Ken-iti. On products of quantities as distributions. Progr. Theoret. Phys. 13, 112–114 (1955).

Noting the well-known difficulty of extending the usual notion of product of functions to the case of distributions in the sense of Schwartz, the author sketches a possible distribution-theoretic formulation of quantum electrodynamics not making explicit use of such products.
I. E. Segal (Chicago, Ill.).

Suura, Hiroshi. On a treatment of many-particle systems in quantum field theory. Progr. Theoret. Phys. 12, 49–71 (1954).

In an attempt to improve the existing meson theory, the author proposes a formalism in which one starts with the distribution of the virtual particles in the true vacuum and around a single real particle, and from these one builds up a complete, but not orthogonal, set of states each of which can be interpreted as representing a prescribed number of real particles. *N. Rosen* (Haifa).

Ortiz Fornaguera, R. On some general properties of static solutions of Schiff's equation. Nuovo Cimento (10) 1, 132–158 (1955).

The static solutions of the non-linear meson equation proposed by L. I. Schiff [Phys. Rev. (2) 84, 1–9, 10–11 (1951)] are investigated. Some of their general properties are derived, and an existence proof is given. Expressions are also obtained for the many-body forces which are present in the Schiff theory. *N. Rosen* (Haifa).

Takahashi, Y. A note concerning the quantization of spinor fields. Nuovo Cimento (10) 1, 414–426 (1955).

The author discusses various methods of quantizing a field with spin 1/2. The Lagrangian used in the discussion differs from that usually considered in the derivation of the Dirac equation. *A. H. Taub* (Urbana, Ill.).

Rohrlich, F. Infrared divergence in bound state problems. Phys. Rev. (2) 98, 181–182 (1955).

In an earlier paper [Jauch and Rohrlich, Helv. Phys. Acta 27, 613–636 (1954); MR 16, 979] it was demonstrated that all infra-red divergences exactly cancel in the iteration solution, in the interaction-representation picture. When the same physical picture is used for certain bound-state problems, the argument seems to fail. In this note it is shown that the apparent difficulty is quite simply resolved, if (taking the case of positronium for example) the bound-interaction picture is employed, where Coulomb wave functions take the place of non-transverse photons.
A. Salam (Cambridge, England).

Edwards, S. F. The nucleon Green function in charged meson theory. Proc. Roy. Soc. London. Ser. A. 228, 411–424 (1955).

In a previous paper [Edwards and Peierls, same Proc. 224, 24–33 (1954); MR 15, 1010] the method of functional integration was used to obtain explicitly the nucleon Green function in neutral scalar theory without recoil. In this paper neutral mesons are replaced by charged mesons, while recoil is still neglected. As an auxiliary step in the calculation, the nucleon Green function in an external charged meson field is required. In contrast to the neutral case, the functional integral representing this Green function cannot be evaluated explicitly. A general method for an approximate evaluation is proposed and it is demonstrated that the method works for the both weak and strong coupling cases. The details are unfortunately too long to be summarized
A. Salam (Cambridge, England).

Itabashi, Kiyomi. On the renormalization in Tamm-Dancoff approximation for one-nucleon problem. II. Subtraction of divergences in the generalized Tamm-Dancoff equations. Progr. Theoret. Phys. 12, 585–602 (1954).

The considerations of an earlier paper [same journal 12, 494–502 (1954)] are extended to formulate an unambiguous procedure for subtracting divergences from the solutions of generalized Tamm-Dancoff equations. In particular, the problem of overlapping divergences is considered.
A. Salam (Cambridge, England).

Ôkubo, Susumu. Diagonalization of Hamiltonian and Tamm-Dancoff equation. Progr. Theoret. Phys. 12, 603–622 (1954).

A powerful formulation for directly diagonalizing the total Hamiltonian in field theory is presented. The results obtained previously by using Tamm-Dancoff type of approximation method are rederived. The normalization problem for the Tamm-Dancoff amplitude is carefully considered.
A. Salam (Cambridge, England).

Caianiello, E. R. Remarks on the existence of derivatives of propagation kernels with respect to the interaction strength. Nuovo Cimento (10) 1, 337–340 (1955).

Burton, W. K. Equivalence of the Lagrangian formulations of quantum field theory due to Feynman and Schwinger. Nuovo Cimento (10) 1, 355–357 (1955).

Milford, F. J. Projection operator for the Rarita-Schwinger equation. Phys. Rev. (2) 98, 1488 (1955).

Stueckelberg, E. C. G., et Wanders, G. Acausalité de l'interaction non-locale. Helv. Phys. Acta 27, 667–682 (1954).

The authors investigate the possibility of a unitary and causal transition, or $S[V]$, matrix in quantized field theory, containing a form factor, i.e., corresponding to the point interaction being replaced by an extended interaction. They conclude that in this case it is not possible to satisfy the causality requirements. *N. Rosen* (Haifa).

Hara, Osamu, Marumori, Toshio, Ohnuki, Yosio, and Shimodaira, Hajime. An attempt to the unified description of elementary particles. Progr. Theoret. Phys. 12, 177–207 (1954).

The work is based on the use of the non-local field of H. Yukawa [Phys. Rev. (2) 77, 219–226 (1950); MR 11, 567], regarded as describing particles with internal structure. States of the internal motion are classified by means of the eigenvalues of commuting variables, two of them being taken as the spin and mass operators. Each such state corresponds to an elementary particle. Scalar and spinor fields are discussed. *N. Rosen* (Haifa).

Giambiagi, J. J., and Tiomno, J. Non-relativistic equation for particles with spin 1. An. Acad. Brasil. Ci. **26**, 327–334 (1954).

The method of L. Foldy and S. A. Wouthuysen [Phys. Rev. (2) **78**, 29–36 (1950)] is applied to the equation for particles of spin 1. See the work of K. M. Case [ibid. **95**, 1323–1328 (1954); MR 16, 656] which deals with the same problem and arrives at similar results. *N. Rosen.*

Pekar, S. I. Theory of strong coupling of a particle (nucleon) with a meson field. Ž. Eksper. Teoret. Fiz. **27**, 398–410 (1954).

The paper deals with a system consisting of a nucleon strongly coupled to a field of charged and neutral mesons. The method used is similar to that in the theory of "polarons": the high frequencies in the meson field are neglected, so that one assumes that the meson variables change slowly during the motion of the nucleon. This leads to equations determining "self-consistent" states of the nucleon in a potential well produced by the meson field. An approximate Hamiltonian is derived for the system.

N. Rosen (Haifa).

Pekar, S. I. The freely moving nucleon. Ž. Eksper. Teoret. Fiz. **27**, 411–420 (1954). (Russian)

This is a continuation of the paper reviewed above. The approximate wave function and energy are calculated for a system of a nucleon strongly coupled to a field of charged and neutral mesons. The meson-field mass of the nucleon in various isobaric states is calculated, as well as the moment of inertia and the quasi-moment of inertia associated with motion of the charge degrees of freedom (isotopic spin). The interaction between the translational and rotational motion gives a dependence of the field mass on the rotational quantum numbers. *N. Rosen* (Haifa).

Pekar, S. I. Criteria of applicability of the theory of strong coupling of particles with a meson field. Ž. Eksper. Teoret. Fiz. **27**, 579–589 (1954). (Russian)

In connection with the author's two papers reviewed above on a nucleon strongly coupled to a meson field, the conditions are set up, in the form of inequalities, for the validity of the approximation in which the nucleon is regarded as being in a discrete energy level in a self-consistent meson-field potential well and moving translationally and rotationally together with this well. *N. Rosen* (Haifa).

Schönberg, M. Vortex motions of the Madelung fluid. Nuovo Cimento (10) **1**, 543–580 (1955).

If a function ψ, satisfying the Schrödinger equation is written as

$$(1) \qquad \psi = R \exp (ik^{-1}S)$$

and if we define

$$(2) \qquad m\mathbf{v} = \operatorname{grad} S - ec^{-1}\mathbf{A}, \quad \rho = R^2,$$

where **A** is the vector potential appearing in the Schrödinger equation, then it follows that

$$(3) \qquad M\left(\frac{\partial}{\partial \tau} + \mathbf{v} \operatorname{grad}\right)\mathbf{v} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times H\right) + \frac{k^2}{2m} \operatorname{grad} P,$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \mathbf{v}) = 0,$$

where $P = \Delta R/R$, **E** and **H** are the electric and magnetic field vectors respectively and Δ is the Laplace operator. Equations (3) are said to be the equations of motion of the Madelung fluid. This paper is concerned with a discussion of general properties of solutions of these equations and with some particular solutions. Solutions of (3) for v and ρ which do not satisfy (2) are also considered and are said to correspond to a generalized Schrödinger equation. The vector **v** is written in terms of three scalar functions, the Clebsch parameters as

$$\mathbf{v} = \operatorname{grad} S + \lambda \operatorname{grad} \mu$$

and a detailed discussion of these parameters is given.

A. H. Taub (Urbana, Ill.).

***Case, K. M., de Hoffmann, F., and Placzek, G. Introduction to the theory of neutron diffusion. Vol. I. Numerical work by B. Carlson and M. Goldstein.** Los Alamos Scientific Laboratory, Los Alamos, N. M., 1953. viii+174 pp. $1.25 (for sale by the Superintendent of Documents, U. S. Government Printing Office, Washington, D. C.).

The theory of neutron diffusion forms the basis for all theoretical work concerning neutron chain reactions. The purpose of this book is to give a detailed discussion of the general equations of the one-velocity neutron diffusion theory in which it is assumed that the magnitude of the neutron velocity is unchanged on collision. The book is based on a series of lectures by Placzek and has then been worked out by him in collaboration with de Hoffmann and Case. After a short introduction, there follows a part treating propagation in the absence of scattering collisions, containing chapters about streaming in vacuum, and the case of a purely absorbing medium. The next part treats the basis of the one-velocity theory of neutron diffusion, containing chapters about equations for a general medium, the case of a uniform infinite medium with isotropic scattering and applications of the results, obtained for the uniform infinite medium, to the solution of final problems. It is pointed out in the preface that only a small part of the general theory of neutron diffusion has been covered, but still the book may be of great help for the understanding of many of the papers dealing with much more complicated problems in this field. The book contains a large amount of numerical material which has been made available by the Los Alamos computing group. A second volume of the book will follow. *P.-O. Löwdin* (Uppsala).

Moses, Harry E. Exchange scattering in a three-body problem. Phys. Rev. (2) **91**, 185–192 (1953).

The problem of the scattering of an electron by a hydrogen atom having an infinitely heavy nucleus is considered as the prototype for a three-body scattering problem. It is shown that the coefficient corresponding to the exchange scattering behaves like a radially outgoing wave. The basic assumption used by Mott and Massey [The theory of atomic collisions, 2nd ed., Oxford, 1949, Chap. VIII, Secs. 2–4] is thus verified. *P.-O. Löwdin* (Uppsala).

Moses, H. E. The scattering operator and the adiabatic theorem. Nuovo Cimento (10) **1**, 103–131 (1955).

The basis for the quantum-mechanical treatment of the scattering problem is still of actual interest. In the older treatments the fundamental assumptions were formulated in a time-independent way, but more recently several authors have used the time-dependent Schrödinger equation together with an assumption that the perturbation is switched off adiabatically as the time approaches $-\infty$. However, it has been shown by Friedrichs [Comm. Appl.

Math. 1, 361–406 (1948); MR 10, 547] that, if one works with an appropriate class of solutions of the time-dependent equations corresponding to wave packets of the eigenfunctions associated with the continuous spectrum of the total Hamiltonian, adiabatic switch-off of the perturbation is unnecessary.

The purpose of the present paper is to use Friedrichs' approach to derive expressions for "out-going and incoming" eigenfunctions and the scattering operator arising from the time-dependent Schrödinger equation. The use of adiabatic switch-off procedures is found unnecessary and the whole treatment will therefore appear more natural than before. A detailed discussion of the adiabatical approach is included for the sake of completeness.
P.-O. Löwdin (Uppsala).

Moses, H. E. Application of variational principles to scattering problems. Phys. Rev. (2) 96, 519–522 (1954).

After a brief review of the scattering operator formalism, the usual forms of the variational expressions are indicated. Schwinger's and Hulthén-Kohn's variational principles are then adapted to scattering problems in which only part of the scattering potential is small. The paper is written by using a modified form of Dirac's bracket notations.
P. O. Löwdin (Uppsala).

Rubinow, S. I. Variational principle for scattering with tensor forces. Phys. Rev. (2) 98, 183–187 (1955).

The variational principle in differential form for the phase shift δ_l in case of a central-force scattering problem is reconsidered. At first, a new formulation of this principle is given which is slightly simpler than the ones previously in use; the derivation shows also why so many formulations have appeared in the literature. By means of matrix notation the procedure is then generalized in a natural way to scattering with tensor forces, in which case there are three parameters, namely the two phase shifts δ_α, δ_β and the mixture parameter ϵ.
P.-O. Löwdin (Uppsala).

Schwebel, Solomon L. An evaluation of approximation methods for three body scattering problems. Div. Electromag. Res., Inst. Math. Sci., New York Univ. Res. Rep. No. **CX-15**, iv+125 pp. (1954).

In the quantum theory of collision phenomena in many-particle systems, approximate methods must be used to a large extent. In order to test the accuracy of some standard methods, Schwebel has considered some one-dimensional three-body problems which may be solved exactly and which at the same time exhibit features occurring in actual physical problems concerning elastic, exchange and inelastic scattering. The cases of distinguishable particles, identical particles with symmetric perturbation, and identical particles with asymmetric perturbation are treated, and, by comparison with the exact solutions, Schwebel has investigated the accuracy of Born's approximation, the Born expansion, and the Schwinger and Hulthén-Kohn variational principles. The book contains also a valuable bibliography.
P. O. Löwdin (Uppsala).

Swan, P. Solution of many-body scattering problems by Fredholm's equation. Phys. Rev. (2) 96, 1144 (1954).

In the quantum theory of many-body elastic collisions, a second-order linear integro-differential equation is of basic importance. It is shown that this equation may be reduced to a Fredholm equation, for which an analytic solution can be constructed. According to the author's opinion, the Fredholm equation is also easier to solve numerically and, as an illustration of the method, an example from the neutron-deuteron scattering theory is solved by iteration.
P. O. Löwdin (Uppsala).

d'Espagnat, Bernard, and Prentki, Jacques. Possible mathematical formulation of the Gell-Mann model for new particles. Phys. Rev. (2) 99, 328–329 (1955).

Gürsey, F. Connection between Dirac's electron and a classical spinning particle. Phys. Rev. (2) 97, 1712–1713 (1955).

This is a short summary of a paper to be published later establishing a connection between the classical and wave mechanics of a spinning relativistic particle. According to the author the 4-dimensional streamlines satisfying the equation $dx^\mu/ds=\bar{\psi}\gamma^\mu\psi$ coincide exactly with the world lines of relativistic spinning particles.
L. Infeld (Warsaw).

Bingel, Werner, Über die Berechnung von Slater-Integralen. Z. Naturf. 9a, 675–684 (1954).

In Slater's theory for atoms containing many electrons, the energy levels are determined by certain integrals $I(nl)$, $F^k(nl, n'l')$, $G^k(nl, n'l')$. Formulas for special types of these integrals have previously been given in the literature, but Bingel is here presenting them for arbitrary quantum numbers in the case when the radial one-electron functions are expressed as sums of powers times exponentials. A comparison between the values of the Slater integrals obtained theoretically from the self-consistent-field functions and those derived semi-empirically from the experimental term values shows a fairly good agreement between the two sets.
P. O. Löwdin (Uppsala).

Ma, S. T. On the Coulomb and Hulthén potentials. Austral. J. Phys. 7, 365–372 (1954).

The Coulomb potential energy $V=-Ze^2/r$ may be considered as a limiting case of the Hulthén potential
$$-Ze^2\cdot\mu e^{-\mu r}/(1-e^{-\mu r})\quad(\mu>0),$$
when $\mu\to 0$. The Hulthén potential has the advantage that the corresponding wave equation may be solved in exact forms. Ma discusses the solutions of the wave equation for the Hulthén potential and considers in detail the limiting process when μ goes to zero. In the second part of the paper, he applies the Fredholm theory to the case of the Hulthén potential, and compares the zeros of the Fredholm determinant as calculated from the first two terms of its power series expansion with the exact values. In a note added in proof, some connections with recent work on the same problem by other authors are given.
P.-O. Löwdin.

Elton, I. R. B. On the infinity in the second Born approximation for the Coulomb field. Proc. Cambridge Philos. Soc. 51, 333–343 (1955).

It is well known that many of the conventional methods in the general theory of scattering do not immediately apply to the case of a Coulomb potential because it decreases too slowly when the distance increases. In order to use for instance the successive Born approximations, it is therefore necessary to consider the Coulomb potential as the limit of a potential which, until the limiting process is applied, decreases more rapidly than $1/r$. Dalitz [Proc. Roy. Soc. London. Ser. A. 206, 509–520 (1951); MR 13, 96] has treated the Coulomb scattering of electrons and positrons by starting from the potential $(\lambda/r)\exp(-\mu r)$ and then

letting μ tend to 0. In the second Born approximation, there is a difficulty arising from the fact that the scattering intensity has a logarithmic infinity proportional to $\ln \mu$, which Dalitz suspects depends on an immaterial phase factor. Elton confirms this suspicion and shows the relationship between this infinity and the immaterial phase factor and the well known distortion at infinity of the Coulomb wave. The treatment is carried out in both the nonrelativistic and relativistic cases. Ways of subtracting out this infinity, which do not lead to any observable effects, are indicated. *P.-O. Löwdin* (Uppsala).

Votruba, Václav, and Lokajíček, Miloš. Nucleon isobars and pion scattering. Czechoslovak J. Phys. 4, 1–13 (1954). (Russian summary)

On the basis of the apparent occurrence of an isobaric state of the nucleon in recent pion-nucleon scattering experiments, the authors assume that the nucleon is a Dirac particle, the isotopic spin vector of which can have two different lengths, 1/2 and 3/2. A convenient set of matrices is introduced to describe this situation and the algebraic relations between them are given. They are further investigated in the paper reviewed hereafter. The rest of the present paper discusses briefly the coupling between nucleon and pion and its implications for pion-nucleon scattering.
L. Van Hove (Utrecht).

Votruba, V., und Christov, Christo Janko. Die Algebra des isotopen Spins ($\frac{3}{2}$, $\frac{1}{2}$). Czechoslovak J. Phys. 4, 403–418 (1954). (Russian summary)

All irreducible hermitian matrix representations are determined for a system of relations established previously by Votruba and Lokajíček for special matrices encountered in the combination of representations $D_{1/2}$ and $D_{3/2}$ of the three-dimensional rotation group [see the paper reviewed above]. *L. Van Hove* (Utrecht).

Munschy, G. Étude d'une famille de fonctions d'onde approchées pour l'atome d'hélium. J. Phys. Radium (8) 16, 473–479 (1955).

The method developed by Pluvinage [Ann. Physique (12) 5, 145–152 (1950); J. Phys. Radium (8) 12, 789–792 (1951); MR 12, 152; 13, 892] is extended so as to apply to higher energy levels. This method is also modified, in that the function u_k is chosen so as to insure a well-behaved solution near the pole, $r_{12}=0$, of the interaction potential. This improvement is especially important for the computation of the lowest energy level. The author also compares calculated and observed energy levels. *A. Erdélyi*.

Thermodynamics, Statistical Mechanics

Popov, Kiril. The mathematical foundations of the theory of irreversible thermodynamic processes. Ž. Eksper. Teoret. Fiz. 28, 257–282 (1955). (Russian)

This development, limited to the linear phenomenological relations of irreversible thermodynamics, is based on the proposition that the thermodynamics flux as the first time derivative, $\dot{x}_i$, of a parameter measuring a departure from equilibrium can be related to the thermodynamic force, X_i, by the equation, $\dot{x}_i=X_i$, where the X_i are linear functions of the x_i. This proposition derives from a stated continuity of entropy as dependent on the departure from equilibrium. From this basis the Onsager reciprocal relations and other characteristics of the linear phenomenological relations are derived by an examination of the solutions of the proposed differential equations. Certain established applications are reviewed. *N. A. Hall* (New York, N. Y.).

Karanikolov, Hristo. On the phenomenological relations of Onsager. Ž. Eksper. Teoret. Fiz. 28, 283–286 (1955). (Russian)

The methods of the paper reviewed above are applied to the simple case of three independent thermodynamic fluxes.
N. A. Hall (New York, N. Y.).

de Groot, S. R., and van Kampen, N. G. On the derivation of reciprocal relations between irreversible processes. Physica 21, 39–47 (1955).

The paper deals with the often discussed question of deriving Onsager's reciprocity relations in the thermodynamics of irreversible processes. It is especially concerned with the derivation for the case of vectorial and tensorial processes, and adopts as starting point the derivation for the scalar case carried out by van Kampen in quantum statistics [Physica 20, 603–622 (1954); MR 16, 322]. Two methods are proposed, differing by their formal aspect rather than by their physical contents. The first uses the results of van Kampen to derive a formula concerning fluctuations, from which point on an earlier derivation by de Groot and Mazur can be taken over [Phys. Rev. (2) 94, 218–224, 224–226 (1954); MR 15, 921]. The second method directly relates the vectorial and tensorial cases to the scalar case by means of a suitable mathematical artifice. It is illustrated on various examples. *L. Van Hove* (Utrecht).

Kluitenberg, G. A., and de Groot, S. R. Relativistic thermodynamics of irreversible processes. IV. Systems with polarization and magnetization in an electromagnetic field. Physica 21, 148–168 (1955).

[Parts I–III appeared in Physica 19, 689–704, 1079–1094 (1953); 20, 199–209 (1954); MR 15, 490; 16, 185, 186.] The authors now discuss irreversible processes in media which are polarizable and magnetizable, the media being isotropic in this respect. They deal with the relativistic second law of thermodynamics, ponderomotive force, entropy balance, phenomenological equations and Onsager's relations.
J. L. Synge (Dublin).

Kluitenberg, G. A., and de Groot, S. R. Relativistic thermodynamics of irreversible processes. V. The energy-momentum tensor of the macroscopic electromagnetic field, the macroscopic forces acting on the matter and the first and second laws of thermodynamics. Physica 21, 169–192 (1955).

Continuing with media which are polarizable and magnetizable (see the preceding review), the authors are led to a symmetric energy tensor $W_{(t)\alpha\beta}=W_{\alpha\beta}+W_{(f)\alpha\beta}$, where $W_{\alpha\beta}$ is the energy tensor of matter and $W_{(f)\alpha\beta}$ that of the electromagnetic field; the conservation equation $\partial W_{(t)\alpha\beta}/\partial x_\beta=0$ is satisfied. Here

$$W_{(f)\alpha\beta}=-B_{\alpha\gamma}H_{\gamma\beta}-\tfrac{1}{4}\delta_{\alpha\beta}B_{\gamma\xi}B_{\gamma\xi}+u_\xi(B_{\xi\gamma}H_{\gamma\alpha}-H_{\xi\gamma}B_{\gamma\alpha})u_\beta,$$

where $B_{\alpha\beta}$ is the tensor formed from **B** and **E** and $H_{\alpha\beta}$ the tensor formed from **H** and **D**; u_α is the 4-velocity. This differs from Abraham's tensor only in the second term (i.e. in the diagonal elements). The thermodynamic formalism is not altered if Abraham's tensor is used instead of $W_{(f)\alpha\beta}$. Minkowski's energy tensor is also considered; but its asymmetry leads to difficulties, because it is desirable to have $W_{(t)\alpha\beta}$ symmetric for the conservation of angular momentum, and the symmetry of $W_{\alpha\beta}$ is a basic assumption in the authors' thermodynamics. *J. L. Synge* (Dublin).

Biot, M. A. Variational principles in irreversible thermodynamics with application to viscoelasticity. Phys. Rev. (2) **97**, 1463–1469 (1955).

This paper purports to give a general formulation of irreversible thermodynamics. The author does not state the assumptions, which appear to multiply as the discussion proceeds. As far as the reviewer can understand the author's intentions, the generalized forces Q_i are assumed related to the generalized state variables q_i by

$$(*)\qquad Q_i = \sum a_{ij} q_j + \sum b_{ij} \dot{q}_j,$$

where the matrices a and b are symmetric. Hence it is obvious that there are quadratic forms analogous to the potential energy and the dissipation function in classical dynamics. The equation (*) is solved operationally. The author proves an extremal theorem to the effect that the velocities $\dot{q}_i$ for systems satisfying (*) render the rate of entropy production a minimum in the class for which a certain power is constant. As an application he derives an integro-differential equation in the linearized theory of visco-elasticity. *C. Truesdell* (Bloomington, Ind.).

Gamba, A. Thermodynamics and quantum mechanics. Nuovo Cimento (10) **1**, 358–360 (1955).

Temkin, A. G. Influence of the integral criterion of form on the process of heat conduction. Ž. Tehn. Fiz. **25**, 497–511 (1955). (Russian)

A certain criterion of form of bodies is defined as $E_s = S^{1/2} V^{-1/3}$, where S is the area of the surface and V is the volume of a given bounded body. For a cylindrical body with one infinite dimension, a similar criterion is defined as $E_s = pF^{-1/2}$, where p is the perimeter while F is the area of the cross-section perpendicular to the axis of the body. The rate of cooling of certain bodies is expressed as a function of these criteria. The object of the work is to give a precise meaning to a general law of cooling which states that the rate of cooling of a given body increases with an increase of the ratio of the area of its surface to the volume of the body.
H. P. Thielman (Ames, Iowa).

Temkin, A. G. A theorem on the maximum of a temperature gradient. Ž. Tehn. Fiz. **25**, 534–540 (1955). (Russian)

This paper is a further development of the ideas presented by the author in the paper reviewed above. It is shown that if the surface of a body is increased without a change in the volume of the body, then there takes place an increase of the rate of heat flow from the body simultaneously with a decrease in the specific heat flow through the surface of the body. *H. P. Thielman* (Ames, Iowa).

Jancel, Raymond. Sur l'hypothèse fondamentale de la mécanique statistique quantique. C. R. Acad. Sci. Paris **240**, 1864–1866 (1955).

Jancel, Raymond. Sur la théorie ergodique en mécanique quantique. C. R. Acad. Sci. Paris **240**, 1693–1695 (1955).

Blanc-Lapierre, André, et Tortrat, Albert. Sur la réduction de certains problèmes fondamentaux de la mécanique statistique à des problèmes classiques du calcul des probabilités. C. R. Acad. Sci. Paris **240**, 2115–2117 (1955).

The authors formulate carefully the reduction of two problems of statistical mechanics to classical problems of probability theory, and indicate methods of solution. I. The total system energy is specified, and the statistical properties of a specified component are to be studied. II. The components of the system are identical, and the number of components in a given state, for specified total energy and number of components, is to be studied. *J. L. Doob.*

Stratonovič, R. L. Entropy of systems with a random number of particles. Ž. Eksper. Teoret. Fiz. **28**, 409–421 (1955). (Russian)

On the basis of a representation of entropy and of the indeterminancy principle the amount of entropy in a volume containing a random number of particles is found.
Author's summary.

Morgenstern, Dietrich. Analytical studies related to the Maxwell-Boltzmann equation. J. Rational Mech. Anal. **4**, 533–555 (1955).

The author studies an integro-differential equation obtained from the Maxwell-Boltzmann equation of the kinetic theory of gases by introduction in the collision integral of a smearing out factor for the distance between the colliding molecules. Whereas this modification is expected not to affect appreciably the physical content of the equation, it has the advantage of making the equation mathematically less singular and thus more amenable for the proof of existence and uniqueness theorems. Such theorems are derived in Section I for a suitable type of differential-functional equation in Banach space. They are applied to the modified Maxwell-Boltzmann equation in Section II, under an assumption of boundedness for the collision cross-section. Section III provides the proof of the trend toward equilibrium in the spatially homogeneous case.
L. Van Hove (Utrecht).

Chisholm, J. S. R., and de Borde, A. H. A new derivation of the fundamental formulae in Fowlerian statistical mechanics. Proc. Cambridge Philos. Soc. **51**, 526–528 (1955).

Whereas Fowler [Statistical mechanics, 2nd ed., Cambridge, 1936] introduces a generating function to change restricted sums into unrestricted sums, this paper uses an integral representation of the δ-function. Everything else appears to be the same. *G. Newell.*

Yuhnovskiĭ, I. R. Binary distribution function for systems of interacting charged particles. I. Ž. Eksper. Teoret. Fiz. **27**, 690–698 (1954). (Russian)

The equation of N. N. Bogolyubov [Problems of dynamical theory in statistical physics, Gostehizdat, Moscow-Leningrad, 1946; MR **13**, 196] for the distribution functions in a neutral system of charged particles is solved for the binary distribution function to various approximations by means of an expansion in powers of a small parameter. The Coulomb potential is replaced by one which is finite at the origin, in order to eliminate divergence difficulties.
N. Rosen (Haifa).

Butler, S. T., and Friedman, M. H. Partition function for a system of interacting Bose-Einstein particles. Phys. Rev. (2) **98**, 287–293 (1955).

A new approximation method is proposed for calculating the partition function of a system of interacting Bose particles. In the partition function the exponential of the hamiltonian is replaced by a product of a very large number of factors close to one, and it is then shown to take the form of a product of effective interaction factors, simply expressed

in terms of the intermolecular potential, on which a large number of convolutions with narrow gaussians must be performed. Expanding the interaction factors to second order, one obtains an expression containing two-body terms as well as three-body terms. The latter are then neglected. This step is justified by the authors on the ground that the uniform distribution of the particles reduces on the average each three-body term to a value much smaller than the two-body terms; the authors unfortunately disregard that the three-body terms are also much more numerous, so that their discussion is not sufficient to ascertain the merit of the approximation. The approximation once made, however, the resulting expression for the partition function becomes tractable for numerical computation of quantum effects. Some preliminary results are reported for the intermolecular potential of helium. *L. Van Hove* (Utrecht).

Friedman, M. H., and Butler, S. T. Bose-Einstein condensation of an imperfect gas. Phys. Rev. (2) 98, 294–299 (1955).

The approximate method developed in the paper reviewed above for determining quantum effects in the partition function of a Bose gas is here applied to the case of hard spheres of radius small compared to the average de Broglie wave length. Beyond the approximations involved in the previous paper further drastic neglections are made, the effects of which are difficult to estimate. They lead to a final formula of simple structure from which thermodynamical functions can be readily calculated. The interesting result is obtained that the specific heat C_v, while continuous in the temperature, has a discontinuity in its slope for a temperature lower than the condensation temperature of an ideal gas of the same density. *L. Van Hove* (Utrecht).

Morrey, Charles B., Jr. On the derivation of the equations of hydrodynamics from statistical mechanics. Comm. Pure Appl. Math. 8, 279–326 (1955).

This paper contains the detailed mathematical derivation of results published earlier by the author under the same title [Proc. Nat. Acad. Sci. U. S. A. 40, 317–322 (1954); MR 16, 890]. *L. Van Hove* (Utrecht).

Fišer, I. Z. On stability of a homogeneous phase. I. General theory. Ž. Eksper. Teoret. Fiz. 28, 171–180 (1955). (Russian)

There are obtained necessary conditions of thermodynamic stability and sufficient conditions of thermodynamic instability of a homogeneous phase in terms of the theory of the radial distribution function. The problem of determining the distribution is correctly formulated. *Author's summary.*

Fišer, I. Z. On stability of a homogeneous phase. II. Determination of the limit of stability. Ž. Eksper. Teoret. Fiz. 28, 437–446 (1955). (Russian)

On the basis of the general theory developed in part I there are given criteria of stability of a homogeneous phase. An analysis is carried out of the basic properties of systems near limit points of stability. Two examples are considered. *Author's summary.*

Fišer, I. Z. On stability of a homogeneous phase. III. The theory of the crystallization curve. Ž. Eksper. Teoret. Fiz. 28, 447–451 (1955). (Russian)

Barker, J. A. The cell theory of liquids. Proc. Roy. Soc. London. Ser. A. 230, 390–398 (1955).

The classical Gibbs phase integral is rewritten exactly in terms of a cell model in which the volume of each cell is taken to be specific volume per molecule. Numerical estimates are calculated for the errors involved in certain approximations. First, in calculating the configuration integral when there is one particle in each cell, an estimate is made of the influence of correlation between the positions of particles in neighboring cells and it is suggested that this can be corrected by making a virial-type expansion. Then for this same integral an estimate is made of the effects of averaging out certain angular dependence of the potential field within the cell. Finally some calculations are made for situations in which there are two particles in a cell but neglecting interaction between cells which the author considers unimportant at high densities. Estimates were made using a Lennard-Jones potential between particles. *G. Newell* (Providence, R. I.).

Potts, R. B. Combinatorial solution of the triangular Ising lattice. Proc. Phys. Soc. Sect. A. 68, 145–148 (1955).

The determinantal method of Kac and Ward for the solution of two-dimensional Ising problems is applied to the triangular lattice. A determinant is obtained whose terms correspond to the terms of the square of the partition function and this determinant is evaluated by a unitary transformation. *F. J. Murray* (New York, N. Y.).

Rushbrooke, G. S., and Scoins, H. I. On the Ising problem and Mayer's cluster sums. Proc. Roy. Soc. London. Ser. A. 230, 74–90 (1955).

This paper reviews and extends work pertaining to the interrelation between the lattice gas and the Ising problems of binary mixtures and ferromagnets with particular emphasis on the use of the Mayer cluster integral formalism for treatment of the Ising problem. It is shown that except for clusters involving only one or two neighboring points, all clusters are multiply connected by the bonds joining neighboring pairs and that the quasi-chemical approximation is equivalent to neglecting the multiply connected clusters. The approximation is carried one step further with no serious complications and resulting critical temperatures are compared with other types of approximations for various lattices. *G. Newell* (Providence, R. I.).

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This volume contains abstracts of the short lectures given at the Congress. These papers will not be reviewed separately. Volumes containing the longer lectures will appear later.

Revue de Mathématiques et de Physique.

Vol. 2 of this journal appeared in 1955. A note in vol. 2 states that vol. 1 appeared under the title: La Science dans la République Populaire Roumaine. Papers are in French, English or German. Another version of this journal appears in Russian under the title: Žurnal Mathematiki i Fiziki. It is a publication of the Académie de la République Populaire Roumaine.

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